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ANALYSIS OF A RETRIAL QUEUE WITH MULTIPLE VACATIONS AND STATE DEPENDENT ARRIVALS

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Abstract. This paper examines an M/G/1 retrial queueing system with multiple vacations and different arrival rates. Whenever the system is empty, the server immediately takes a vacation. At a vacation completion epoch, if the number of customers in the orbit is at least one the server remains in the system to activate service, otherwise the server avails multiple vacations until at least one customer is recorded in the orbit. The primary arrival rate is λ_1 when the server in idle and the primary arrival rate is λ_2 when the server is busy or on vacation ($\lambda_1 > \lambda_2$). The steady state queue size distribution of number of customers in the retrial group, expected number of customers in the retrial group and expected number of customers in the system are obtained. Some special cases are also discussed. Numerical illustrations are also provided.

Keywords. Retrial queue, single server, multiple vacations, state dependent arrivals, generating function, orbit size.

Mathematics Subject Classification. 90B22.

1. Introduction

A retrial queueing system consists of a primary service facility and an orbit. Customers arrive at the service facility either from outside the system or from the orbit. Upon arrival of a customer, if the server is busy or on vacation the arrival

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will join the retrial group in the orbit and request for service again at some later time. Retrial queues have been widely used to model many problems in telephone switching systems.

The detailed overviews of the related references with retrial queues can be found in the book of Falin and Templeton [16] and the survey papers, of Artalejo [3, 4]. The single server retrial queue with priority calls have been studied by Choi *et al.* [12–14] for many applications in telecommunication and mobile communication. Aissani and Artalejo [1] have analyzed single server retrial queue subject to break downs. Artalejo and Gomez-Corral [7] have made a detailed study on retrial queueing systems.Atentia *et al.* [9] have developed an M/G/1 retrial queue with active breakdowns and Bernoulli schedule in the server. And also Efrosinin and Sztrik [15] have done an analysis on the performance of a two server heterogeneous retrial queue with threshold policy. Stochastic analysis of a single server retrial queue with general retrial times was done by Gomez-Correl [17]. Moreno [24] has done an analysis on the retrial queue with recurrent customers and general retrial times. Artalejo [5] has studied the steady state analysis of the M/G/1 queueing system with repeated attempts and two phase service Embedded Markov chain method and generalize both the classical M/G/1 retrial queue and the M/G/1 queue with classical waiting line and second optional service. There are a number of retrial queues in literature, that were investigated under the constant retrial policy, for instance [6, 10].

Server vacation models are useful for the systems in which the server wants to utilize the idle time for different purposes. Over the past two decades, queueing systems with vacations have been studied by many researchers due to their wide application in production inventory systems, communication systems, computer systems, *etc.* A comprehensive and excellent study on the vacation models can be found in Takagi [26]. For related literature of retrial queues with vacations, Li and Yang [23] developed an M/G/1 retrial system with server vacations and M independent identical input sources. Later Artalejo [2] analyzed an M/G/1 retrial queue with exhausted server vacations that is the server takes a vacation only when there are no customers in the orbit. Batch arrival Markovian single server queueing systems with multiple vacations were first studied by Baba [11]. Senthilkumar and Arumuganathan [25] have analyzed single server batch arrival retrial queue with general vacation time under Bernoulli schedule and two phases of heterogeneous service. Lee and Srinivas [20] have studied batch arrival queue with control policy [N-Policy] and vacations. The variations and extensions of these vacation models can be referred to Lee *et al.* [21,22] and Krishna Reddy *et al.* [18]. Later Arumuganathan and Jeyakumar [8] introduced control Policy on request for re-service for a bulk queue with multiple vacations. They also proposed a cost model for a practical situation and how the results would be useful in optimizing the cost. Most of the retrial queueing papers with vacations have analyzed systems with single type of arrival. However the study of retrial queueing system with different arrival rates is interesting and not much work in this direction is found in literature. Based on this observation, we have investigated a retrial queueing system with multiple vacations and state dependent arrivals. Here the arrival rates are different when the system is idle and busy or on vacation.

Our retrial queue has applications in packet-switched network. The router is an inter connection device that attaches two or more networks in a packet-switched network, which takes charge of receiving packets and forwarding them to the next hop, according to the routing information found in its routing table. The packets arrive at the router according to a Poisson stream. A packet receives service immediately if the router is idle or it will queue up in the buffer (retrial group). To keep the router functioning well, some maintenance activities, such as routing information backup and virus scan is performed when the router is idle. Here two types of packets arrive (urgent and regular) to the router. When the router is idle more number of packets is processed, that is the arrival is more and when the system is busy or on vacation the arrival is less. In this scenario, buffers in the router, router retransmission policy and maintenance activities correspond to the queue and orbit, the server, the retrial discipline and the vacation policy respectively.

2. The mathematical model

In this paper an M/G/1 retrial queueing system with multiple vacations and different arrival rates is analyzed. The customers arrive according to Poisson process with different arrival rates. If the server is busy or on vacation at the arrival epoch, the customers join the orbit to repeat its request later, whereas if the server is idle then the arriving customer begins its service immediately. The customers in the orbit try for service one by one with retrial rate 'γ' when the server is idle. Whenever the system is empty, the server immediately takes a vacation. If there is at least one customer found waiting in the queue upon returning from a vacation, the server will be immediately activated for service. The primary arrival rate is λ_1 when the server is idle and the primary arrival rate is λ_2 when the server is busy or on vacation ($\lambda_1 > \lambda_2$).

Let $S(x)(s(x))\,\{\tilde{S}\,(\theta)\}\,[S^0(x)]$ be the cumulative distribution function (probability density function) {Laplace transform} [remaining service time] of service. $V(x)(v(x))\,\{\tilde{V}\,(\theta)\}\,[V^0(x)]$ be the cumulative distribution function (probability density function) {Laplace transform} [remaining vacation time] of vacation. $N(t)$ denotes the number of customers in the orbit at time t.

The server state is denoted as follows.

$$C(t) = \begin{cases} 0, \text{ if the server is idle} \\ 1, \text{ if the server is busy} \\ 2, \text{ if the server is on vacation.} \end{cases}$$

$Y(t) = j$, if the server is on the jth vacation. The system state probabilities are defined as follows.

(1) $P_{0n}(t) = P_r\{N(t) = n, C(t) = 0\}$, $n \geq 1$ is the probability that at time t the server is idle and the orbit size is n.

(2) $P_{1n}(x,t)\mathrm{d}t = P_r\{N(t)=n, C(t)=1, x \le S^0(t) \le x+\mathrm{d}t\}$, $n \ge 0$ is the probability that at time t the server is busy, the orbit size is n and the remaining service time of the customer under service is between x and $x+\mathrm{d}t$.

(3) $V_{jn}(x,t)\mathrm{d}t = P_r\{N(t)=n, Y(t)=j, C(t)=2, x \le V_0(t) \le x+\mathrm{d}t\}$, $n \ge 0$, $j \ge 1$ is the probability that at time t the server is on jth vacation, the orbit size is n and the remaining vacation time of a customer is between x and $x+\mathrm{d}t$.

3. Steady state queue size distribution

To derive the steady state queue size distribution, the following equations are obtained using supplementary variable technique.

$$P_{0,j}(t+\Delta t) = P_{0,j}(t)\,(1-\lambda_1\Delta t - j\gamma\Delta t) + P_{1,j}(0,t)\,\Delta t + \sum_{l=1}^{\infty} V_{l,j}(0,t)\Delta t, \qquad j \ge 1$$

$$P_{1,j}(x-\Delta t, t+\Delta t) = P_{1,j}(x,t)\,(1-\lambda_2\Delta t) + \lambda_2 P_{1,j-1}(x,t)\,\Delta t + (j+1)\gamma P_{0,j+1}(t)\Delta t\, s(x) + \lambda_1 P_{0,j}(t)\, s(x)\,\Delta t, \qquad j \ge 0$$

$$V_{1,0}(x-\Delta t, t+\Delta t) = V_{1,0}(x,t)\,(1-\lambda_2\Delta t) + P_{1,0}(0,t)v(x)\,\Delta t$$

$$V_{1,j}(x-\Delta t, t+\Delta t) = V_{1,j}(x,t)\,(1-\lambda_2\Delta t) + \lambda_2 V_{1,j-1}(x,t)\,\Delta t, \qquad j \ge 1$$

$$V_{l,0}(x-\Delta t, t+\Delta t) = V_{l,0}(x,t)\,(1-\lambda_2\Delta t) + V_{l-1,0}(0,t)v(x)\,\Delta t, \qquad l \ge 2$$

$$V_{l,j}(x-\Delta t, t+\Delta t) = V_{l,j}(x,t)\,(1-\lambda_2\Delta t) + \lambda_2 V_{l,j-1}(x,t)\,\Delta t, \qquad j \ge 1, \quad l \ge 2.$$

The steady state equations of the system is derived as

$$(\lambda + j\gamma)P_{0,j} = P_{1,j}(0) + \sum_{l=1}^{\infty} V_{l,j}(0), \qquad j \ge 1 \tag{3.1}$$

$$-P'_{1,j} = -\lambda_2 P_{1,j}(x) + \lambda_2 P_{1,j-1}(x) + (j+1)\gamma P_{0,j+1}(0)s(x) + \lambda_1 P_{0,j}(0)s(x), \qquad j \ge 0 \tag{3.2}$$

$$-V'_{1,0}(x) = -\lambda_2 V_{1,0}(x) + P_{1,0}(0)v(x), \tag{3.3}$$

$$-V'_{1,j}(x) = -\lambda_2 V_{1,j}(x) + \lambda_2 V_{1,j-1}(x), \qquad j \ge 1 \tag{3.4}$$

$$-V'_{l,0}(x) = -\lambda_2 V_{1,0}(x) + V_{l-1,0}(0)v(x), \qquad l \ge 2 \tag{3.5}$$

$$-V'_{l,j}(x) = -\lambda_2 V_{l,j}(x) + \lambda_2 V_{l,j-1}(x) \qquad j \ge 1. \tag{3.6}$$

The Laplace transforms (LT) of $P_{1,j}(x)$, $V_{l,j}(x)$ are defined as

$$\mathrm{LT}\left(P_{1,j}(x)\right) = \tilde{P}_{1,j}(\theta) = \int_0^\infty \mathrm{e}^{-\theta x} P_{1,j}(x)\,\mathrm{d}x;$$

$$\mathrm{LT}\left(V_{l,j}(x)\right) = \tilde{V}_{l,j}(\theta) = \int_0^\infty \mathrm{e}^{-\theta x} V_{l,j}(x)\,\mathrm{d}x.$$

Taking Laplace transform on steady state equations (3.2)–(3.6), we have

$$\theta\tilde{P}_{1,j}(\theta) - P_{1,j}(0) = \lambda_2\tilde{P}_{1,j}(\theta) - \lambda_2\tilde{P}_{1,j-1}(\theta) - (j+1)\gamma P_{0,j+1}(0)\tilde{S}(\theta) - \lambda_1 P_{0,j}(0)\tilde{S}(\theta) \qquad j \geq 1 \tag{3.7}$$

$$\theta\tilde{V}_{1,0}(\theta) - V_{1,0}(0) = \lambda_2\tilde{V}_{1,0}(\theta) - P_{1,0}(0)\tilde{V}(\theta) \tag{3.8}$$

$$\theta\tilde{V}_{1,j}(\theta) - V_{1,j}(0) = \lambda_2\tilde{V}_{1,j}(\theta) - \lambda_2\tilde{V}_{1,j-1}(\theta), \qquad j \geq 1 \tag{3.9}$$

$$\theta\tilde{V}_{l,0}(\theta) - V_{l,0}(0) = \lambda_2\tilde{V}_{l,0}(\theta) - V_{l-1,0}(0)\tilde{V}(\theta), \quad l \geq 2 \tag{3.10}$$

$$\theta\tilde{V}_{l,j}(\theta) - V_{l,j}(0) = \lambda_2\tilde{V}_{l,j}(\theta) - \lambda_2\tilde{V}_{l,j-1}(\theta), \qquad j \geq 1, \quad l \geq 2. \tag{3.11}$$

Lee [19] developed a technique to find the steady state probability generating function (PGF) of the number of customers in the queue at an arbitrary time epoch. To apply the technique, the following probability generating functions are defined.

$$P_0(z) = \sum_{j=1}^{\infty} P_{0,j}(0) z^j$$

$$\tilde{P}_1(z,\theta) = \sum_{j=0}^{\infty} \tilde{P}_{1,j}(\theta) z^j; \qquad P_1(z,0) = \sum_{j=0}^{\infty} P_{1,j}(0) z^j;$$

$$\tilde{V}_l(z,\theta) = \sum_{j=0}^{\infty} \tilde{V}_{l,j}(\theta) z^j; \qquad V_l(z,0) = \sum_{j=0}^{\infty} V_{l,j}(0) z^j \quad \text{where} \quad |z| \leq 1. \tag{3.12}$$

Multiplying equations (3.1), (3.8) and (3.9) by Z^0, equations (3.7), (3.9) and (3.11) by Z^n, taking summation from $n = 0$ to ∞ and using (3.12), we get

$$\lambda_1 P_0(z) + \gamma z P_0'(z) = P_1(z,0) - P_{1,0}(0) + \sum_{l=1}^{\infty}\left(V_l(z,0) - V_{l,0}(0)\right), \tag{3.13}$$

$$\left(\theta - \lambda_2 + \lambda_2 z\right)\tilde{P}_1(z,\theta) = P_1(z,0) - \gamma P_0'(z)\tilde{S}(\theta) - \lambda_1 P_0(z)\tilde{S}(\theta), \tag{3.14}$$

$$\left(\theta - \lambda_2 + \lambda_2 z\right)\tilde{V}_1(z,\theta) = V_1(z,0) - P_{1,0}(0)\tilde{V}(\theta), \tag{3.15}$$

$$\left(\theta - \lambda_2 + \lambda_2 z\right)\tilde{V}_l(z,\theta) = V_l(z,0) - V_{l-1,0}(0)\tilde{V}(\theta). \tag{3.16}$$

3.1. Probability generating function of the orbit size

Theorem 3.1. *The probability generating function $P(z)$ of number of customers in orbit is given by*

$$P(z) = \frac{P_0(z)\left[\lambda_2\left(z - \tilde{S}\left(\lambda_2 - \lambda_2 z\right)\right) + \lambda_1\left(\tilde{S}\left(\lambda_2 - \lambda_2 z\right) - 1\right)\right]}{\lambda_2\left[z - \tilde{S}\left(\lambda_2 - \lambda_2 z\right)\right]}$$

$$+ \frac{\left[\tilde{V}\left(\lambda_2 - \lambda_2 z\right) - 1\right]\left[\sum\limits_{l=1}^{\infty} V_{l,0}(0) + P_{1,0}(0)\right]}{\lambda_2\left[z - \tilde{S}\left(\lambda_2 - \lambda_2 z\right)\right]},$$

where

$$P_0(z) = P_0(1)K(z) + K(z)\int\limits_1^z \frac{g(t)}{K(t)\gamma\left[t - \tilde{S}\left(\lambda_2 - \lambda_2 t\right)\right]}\mathrm{d}t,$$

and

$$K(z) = \exp\left[-\frac{\lambda_1}{\gamma}\int_1^z \left[\frac{\tilde{S}\left(\lambda_2 - \lambda_2 u\right) - 1}{u - \tilde{S}\left(\lambda_2 - \lambda_2 u\right)}\right]\mathrm{d}u\right],$$

$$g(z) = \left[\tilde{V}\left(\lambda_2 - \lambda_2 z\right) - 1\right]\left[\sum_{l=1}^{\infty} V_{l,0}(0) + P_{1,0}(0)\right].$$

Proof. The probability generating function $P(z)$ of number of customers in orbit at an arbitrary epoch can be expressed as follows,

$$P(z) = P_0(z) + \tilde{P}_1(z,0) + \sum_{l=1}^{\infty} \tilde{V}_l(z,0) \tag{3.17}$$

Using equation (3.13)–(3.16) we derive the expressions for $P_0(z)$, $\tilde{P}_1(z,0)$, $\tilde{V}_l(z,0)$ as (complete derivation is given in appendix),

$$P_0(z) = P_0(1)K(z) + K(z)\int\limits_1^z \frac{g(t)}{K(t)\gamma\left[t - \tilde{S}\left(\lambda_2 - \lambda_2 t\right)\right]}\mathrm{d}t \tag{3.18}$$

where $K(z) = \exp\left[-\frac{\lambda_1}{\gamma}\int\limits_1^z \left[\frac{\tilde{S}\left(\lambda_2 - \lambda_2 u\right) - 1}{u - \tilde{S}\left(\lambda_2 - \lambda_2 u\right)}\right]\mathrm{d}u\right]$

and $g(z) = \left[\tilde{V}\left(\lambda_2 - \lambda_2 z\right) - 1\right]\left[\sum\limits_{l=1}^{\infty} V_{l,0}(0) + P_{l,0}(0)\right]$.

$$\tilde{P}_1(z,0) = \frac{\left[\tilde{S}\left(\lambda_2 - \lambda_2 z\right) - 1\right]\left[\lambda_1 P_0(z) + \gamma P_0'(z)\right]}{\left(-\lambda_2 + \lambda_2 z\right)}, \tag{3.19}$$

$$\tilde{V}_1(z,0) = \frac{\left[\tilde{V}\left(\lambda_2 - \lambda_2 z\right) - 1\right] P_{1,0}(0)}{\left(-\lambda_2 + \lambda_2 z\right)}, \tag{3.20}$$

$$\tilde{V}_l(z,0) = \frac{\left[\tilde{V}\left(\lambda_2 - \lambda_2 z\right) - 1\right] V_{l-1,0}(0)}{\left(-\lambda_2 + \lambda_2 z\right)}. \qquad l \geq 2 \tag{3.21}$$

Substituting (3.18)–(3.21) in equation (3.17) we get

$$P(z) = \frac{P_0(z)\left[\lambda_2\left(z - \tilde{S}\left(\lambda_2 - \lambda_2 z\right)\right) + \lambda_1\left(\tilde{S}\left(\lambda_2 - \lambda_2 z\right) - 1\right)\right]}{\lambda_2\left[z - \tilde{S}\left(\lambda_2 - \lambda_2 z\right)\right]} + \frac{\left[\tilde{V}\left(\lambda_2 - \lambda_2 z\right) - 1\right]\left[\sum\limits_{l=1}^{\infty} V_{l,0}(0) + P_{1,0}(0)\right]}{\lambda_2\left[z - \tilde{S}\left(\lambda_2 - \lambda_2 z\right)\right]} \tag{3.22}$$

where $P_0(z)$ is given by equation (3.18). □

3.2. Stability condition

The probability generating function $P(z)$ has to satisfy the condition $\lim\limits_{z\to 1} P(z) = 1$. In order to satisfy this condition L'Hospitals rule is applied to equation (3.22). Since $P_{1,0}(0)$ and $\sum_{l=1}^{\infty} V_{l,0}(0)$ are probabilities the numerator of $P(z)$ is positive when $z \to 1$. So $\lim\limits_{z\to 1} P(z) = 1$ is satisfied only if $\rho = \lambda_2 E(s) < 1$. Thus $\rho < 1$ is the condition to be satisfied for the existence of steady state for the model under consideration.

3.3. Computational aspects of unknown constant

In this section the unknown constant is expressed in terms of known constant. Since the model under consideration is an M/G/1 retrial queue it should have only one unknown constant in $P(z)$, but it has two unknowns. So to express the unknown constant namely q_0 in terms of known constant p_0, the following theorem is used.

Theorem 3.2. *If α_n is the probability of n customers arriving during a vacation then, $q_0 = \frac{\alpha_0 p_0}{1-\alpha_0}$ where $q_0 = \sum_{l=1}^{\infty} V_{l,0}(0)$ and $p_0 = p_{1,0}(0)$.*

Proof. Using $\sum_{l=1}^{\infty} V_{l,0} = q_0$, $p_{1,0}(0) = p_0$ equations (A.4), (A.6), simplifies to

$$\begin{aligned}\sum_{l=1}^{\infty} V_l(z,0) &= \sum_{l=1}^{\infty}\sum_{n=0}^{\infty} V_{l,n}(0)z^n\\ &= \tilde{V}\left(\lambda_2 - \lambda_2 z\right)\left[\sum_{l=1}^{\infty} V_{l,0}(0) + P_{1,0}(0)\right]\\ &= \sum_{n=0}^{\infty} \alpha_n z^n \left[q_0 + p_0\right].\end{aligned}$$

Equating the constant terms on both sides of the above equation we have

$$\begin{aligned} q_0 &= \alpha_0\left(p_0 + q_0\right).\\ \therefore q_0 &= \frac{\alpha_0 p_0}{1-\alpha_0}. \end{aligned} \tag{3.23}$$

Hence the theorem. □

4. Performance characteristics

In this section, some useful performance measures of the proposed model like, expected number of customers in the orbit, expected number of customers in the system, probability that the server is idle, probability that the server is busy and the probability that the server is on vacation are derived.

4.1. The mean number of customers in the orbit

The expected number of customers in the orbit is derived using probability generating function (3.22) and $L_Q = E(N(t)) = \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}z}P(z)$,

$$\begin{aligned} L_Q = &\frac{S_2\left[P_0(1)\left(\lambda_2\left(1-\lambda_2 E(s)\right)+\lambda_1\lambda_2 E(s)\right)+\left(q_0+p_0\right)\lambda_2 E(v)\right]}{2\lambda_2\left[1-\lambda_2 E(s)\right]^2}\\ &+\frac{\left[1-\lambda_2 E(s)\right]\left[2P_0'(1)\left(\lambda_2\left(1-\lambda_2 E(s)\right)+\lambda_1\lambda_2 E(s)\right)\right.}{2\lambda_2\left[1-\lambda_2 E(s)\right]^2}\\ &+\frac{\left.P_0(1)\left(-\lambda_2 S_2+\lambda_1 S_2\right)+\left(q_0+p_0\right)V_2\right]}{2\lambda_2\left[1-\lambda_2 E(s)\right]^2}, \end{aligned} \tag{4.1}$$

where

$$S_2 = \lambda_2 E(s) + \lambda_2^2 E\left(s^2\right),$$
$$V_2 = \lambda_2 E(v) + \lambda_2^2 E\left(v^2\right),$$
$$q_0 = \sum_{l=1}^{\infty} V_{l,0}(0), \qquad p_0 = p_{1,0}(0),$$
$$P_0(1) = \frac{1 - \lambda_2 E(s) - E(v)\left(q_0 + p_0\right)}{\left[1 + E(s)\left(\lambda_1 - \lambda_2\right)\right]}, \tag{4.2}$$
$$P_0'(1) = \frac{\lambda_1 \lambda_2 E(s) + \lambda_2 E(v)\left(q_0 + p_0\right)}{\gamma\left[1 + E(s)\left(\lambda_1 - \lambda_2\right)\right]}\cdot \tag{4.3}$$

4.2. Expected number of customers in the system

The probability generating function of the numbers of customers in the system is obtained as follows using equations (3.18)–(3.21).

$$\pi(z) = P_0(z) + Z\left(\tilde{P}_1(z,0)\right) + \sum_{l=1}^{\infty} \tilde{V}_l(z,0),$$
$$= \frac{P_0(z)\left[\lambda_2\left(z - \tilde{S}\left(\lambda_2 - \lambda_2 z\right)\right) + \lambda_1 z\left(\tilde{S}\left(\lambda_2 - \lambda_2 z\right) - 1\right)\right]}{\lambda_2\left[z - \tilde{S}\left(\lambda_2 - \lambda_2 z\right)\right]}$$
$$+ \frac{\left[\tilde{V}\left(\lambda_2 - \lambda_2 z\right) - 1\right]\left[\sum\limits_{l=1}^{\infty} V_{l,0}(0) + P_{1,0}(0)\right]}{\lambda_2\left[z - \tilde{S}\left(\lambda_2 - \lambda_2 z\right)\right]}\cdot \tag{4.4}$$

Using equation (4.4) the mean number of customers in the system is derived as

$$L_s = \lim_{z\to 1} \frac{\mathrm{d}}{\mathrm{d}z}\pi(z)$$
$$= L_Q + \frac{P_0(1)\lambda_1 \rho}{2\left(1 - \rho\right)}, \tag{4.5}$$

where $P_0(1)$ is given by equation (4.2).

4.3. Probability that the server is idle

Let I be the idle period random variable and let $P(I)$ be the probability that the server is idle at time t. Using equation (3.18) and applying $\lim z \to 1$, we get the probability that the server is idle at time t as,

$$P(I) = \frac{1 - \rho - E(v)\left(q_0 + p_0\right)}{\left[1 + E(s)\left(\lambda_1 - \lambda_2\right)\right]}\cdot \tag{4.6}$$

4.4. Probability that the server is busy

Let B be the busy period random variable and $P(B)$ be the probability that the server is busy at time t. Using equation (3.19) and applying $\lim z \to 1$, we get the probability that the server is busy at time t as

$$P(B) = \frac{E(s)\left[\lambda_1 + (\lambda_2 - \lambda_1)\, E(v)\,(q_0 + p_0)\right]}{\left[1 + E(s)\,(\lambda_2 - \lambda_1)\right]}. \tag{4.7}$$

4.5. Probability that the server is on vacation

Let V be the random variable for multiple vacations and $P(V)$ be the probability that the server is on multiple vacations at time t. Using equations (3.20) and (3.21) and applying limit $z \to 1$, we get the probability that the server is on vacation at time t as,

$$P(V) = \frac{E(v)\,(q_0 + p_0)}{\left[1 + E(s)\,(\lambda_1 - \lambda_2)\right]}. \tag{4.8}$$

5. Particular case

In this section, a particular case of the proposed model is presented. If theserver does not avail multiple vacations and arrival rates are same, that is $\tilde{V}(\lambda_2 - \lambda_2 z) = 1$ and $\lambda_1 = \lambda_2 = \lambda$, then equation (3.22) reduces to

$$P(z) = \frac{P_0(z)(z-1)}{\left[z - \tilde{S}\,(\lambda_2 - \lambda_2 z)\right]}. \tag{5.1}$$

This equation exactly coincides with the result of orbit size distribution of $\mathrm{M}^x/\mathrm{G}/1$ retrial queueing system by Falin and Templeton [16] when the arrival is not batch arrival.

6. Special cases

Further, by specifying service time random variables as Exponential, Erlang and Hyper Exponential distribution, some special cases of the proposed model are discussed below:

Case (i): Single server retrial queue with Exponential service time, multiple vacations and different arrival rates.

If the service time is assumed to be exponential with probability density function $s(x) = u\mathrm{e}^{-ux}$, where u is the parameter, then

$$\tilde{S}\,(\lambda_2 - \lambda_2 z) = \left(\frac{u}{u + \lambda_2(1-z)}\right). \tag{6.1}$$

Substituting (6.1) in (3.22), the PGF of the retrial queue size distribution for the single server retrial queue with multiple vacations and different arrival rates is given by

$$P(z) = \frac{P_0(z)\left[\lambda_2\left\{z-\left(u/\left(u+\lambda_2(1-z)\right)\right)\right\}+\lambda_1\left\{\left(u/\left(u+\lambda_2(1-z)\right)\right)-1\right\}\right]}{\lambda_2\left[z-\left(u/\left(u+\lambda_2(1-z)\right)\right)\right]} + \frac{\left[\tilde{V}\left(\lambda_2-\lambda_2 z\right)-1\right]\left[\sum_{l=0}^{\infty} V_{l,0}(0)+P_{1,0}(0)\right]}{\lambda_2\left[z-\left(u/\left(u+\lambda_2(1-z)\right)\right)\right]}, \tag{6.2}$$

where

$$P_0(z) = P_0(1)K(z) + K(z)\int_1^z \frac{g(t)}{K(t)\gamma\left[t-\left(u/\left(u+\lambda_2(1-t)\right)\right)\right]}\mathrm{d}t,$$

$$K(z) = \exp\left\{-\frac{\lambda_1}{\gamma}\int_1^z\left[\frac{\left(u/\left(\left(u+\lambda_2(1-z)\right)-1\right)\right)}{z-\left(u/\left(u+\lambda_2(1-z)\right)\right)}\right]\mathrm{d}z\right\},$$

and $g(z) = \left[\tilde{V}\left(\lambda_2-\lambda_2 z\right)-1\right]\left[\sum_{l=1}^{\infty} V_{l,0}(0)+P_{1,0}(0)\right]$.

Case (ii): Single server retrial queue with Erlang service time, multiple vacations and different arrival rates.

If the service time is assumed to be Erlang with probability density function $s(x) = \frac{(k\,u)^k x^{k-1}\mathrm{e}^{-k\,u\,x}}{(k-1)!}$, $k > 0$ where u is the parameter then,

$$\tilde{S}\left(\lambda_2-\lambda_2 z\right) = \left[u\,k/\left(u\,k+\left(\lambda_2-\lambda_2 z\right)\right)\right]^k. \tag{6.3}$$

Substituting (6.3) in (3.22), the PGF of the retrial queue size distribution for single server retrial queue with multiple vacations and different arrival rates is given by

$$P(z) = \frac{P_0(z)\left[\lambda_2\left\{z-\left[u\,k/(u\,k+(\lambda_2-\lambda_2 z))\right]^k\right\}+\lambda_1\left\{\left[u\,k/(u\,k+(\lambda_2-\lambda_2 z))\right]^k-1\right\}\right]}{\lambda_2\left\{z-\left[u\,k/\left(u\,k+\left(\lambda_2-\lambda_2 z\right)\right)\right]^k\right\}} + \frac{\left[\tilde{V}\left(\lambda_2-\lambda_2 z\right)-1\right]\left[\sum_{l=0}^{\infty} V_{l,0}(0)+P_{l,0}(0)\right]}{\lambda_2\left\{z-\left[u\,k/\left(u\,k+\left(\lambda_2-\lambda_2 z\right)\right)\right]^k\right\}}. \tag{6.4}$$

Case (iii): Single server retrial queue with Hyper Exponential service time, multiple vacations and different arrival rates.

If the service time is assumed to be Hyper Exponential with probability density function $s(x) = c\,u\,\mathrm{e}^{-u\,x} + (1-c)c\,u\,\mathrm{e}^{-w\,x}$ then

$$\tilde{S}\left(\lambda_2-\lambda_2 z\right) = \left[u\,c/\left(u+\left(\lambda_2-\lambda_2 z\right)\right)\right] + \left[w(1-c)/\left(w+\left(\lambda_2-\lambda_2 z\right)\right)\right]. \tag{6.5}$$

Substituting (6.5) in (3.22), the PGF of the retrial queue size distribution for single server retrial queue with multiple vacations and different arrival rates is obtained.

TABLE 1. Retrial rate γ verses mean orbit size L_Q. ($\lambda_1 = .8$, $\lambda_2 = .4$).

γ	Exponential L_Q	Erlang-2 L_Q	Hyper exponential L_Q
1	0.7081	0.8469	0.8820
2	0.4575	0.5345	0.5586
3	0.3740	0.4304	0.4508
4	0.3322	0.3783	0.3969
5	0.3072	0.3471	0.3646
6	0.2905	0.3263	0.3430
7	0.2785	0.3114	0.3276
8	0.2696	0.3002	0.3161
9	0.2626	0.2915	0.3071
10	0.2570	0.2846	0.2999

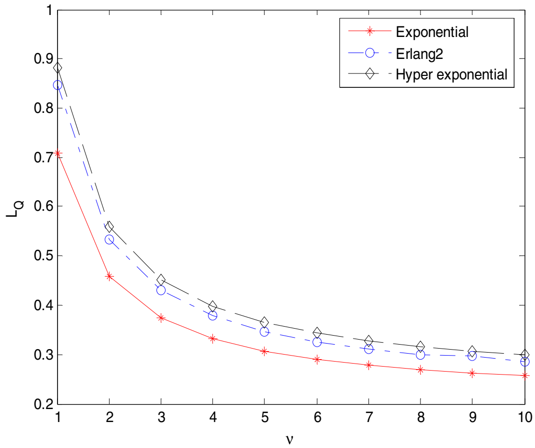

FIGURE 1. Retrial rate γ verses mean orbit size L_Q.

7. NUMERICAL RESULTS

In this section, to justify the theoretical results obtained, we present some numerical results. Here the effects of the arrival rates λ_1, λ_2 and retrial rate γ on the mean orbit size L_Q are analyzed with the following assumptions and notations:

(i) average arrival rate when the system is idle λ_1;
(ii) average arrival rate when the system is busy or on vacation λ_2;
(iii) service rate μ;
(iv) vacation duration is exponential with parameter η;
(v) retrial rate γ.

Table 1 and Figure 1 represent the effect of retrial rate γ on the mean orbit size. The service times are considered as exponential, Erlang-2 and hyper exponential with parameters $\lambda_1 = .8$, $\lambda_2 = .4$, $\mu = 7$ and $\eta = 2$ it is observed that the mean orbit size is decreasing when the retrial rate increases.

TABLE 2. Arrival rate λ_1 verses mean orbit size L_Q. ($\gamma = 5$, $\lambda_2 = .1$).

λ_1	Exponential	Erlang-2	Hyper exponential
	L_Q	L_Q	L_Q
.2	0.0716	0.0737	0.0746
.3	0.0727	0.0762	0.0777
.4	0.0738	0.0789	0.0809
.5	0.0750	0.0816	0.0842
.6	0.0762	0.0845	0.0877
.7	0.0774	0.0875	0.0913
.8	0.0787	0.0906	0.0951
.9	0.0799	0.0939	0.0989
1.0	0.0812	0.0972	0.1030
1.1	0.0826	0.1007	0.1072

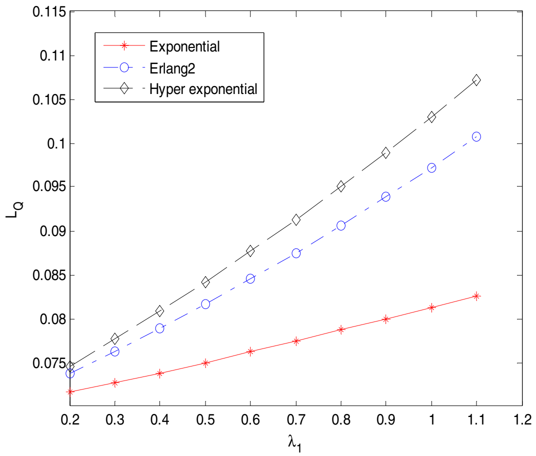

FIGURE 2. Arrival rate λ_1 verses mean orbit size L_Q.

Table 2 and Figure 2 represent the effect of arrival rate λ_1 (when the system is idle). Considering the service times as exponential, Erlang-2 and hyper experiential with parameters $\gamma = 5$, $\mu = 7$, $\eta = 2$ and $\lambda_2 = .1$ it is observed that the mean orbit size is increasing when the arrival rate λ_1 increases.

Table 3 and Figure 3 represent the effect of arrival rate λ_2 (when the server is busy or on vacation) on the mean orbit-size. Considering the service times as exponential, Erlang-2 and hyper exponential with parameters $\gamma = 5$, $\mu = 7$, $\eta = 2$ and $\lambda_1 = 1.1$, it is observed that mean orbit size is increasing when the arrival rate λ_2 increases.

8. CONCLUSION

In this paper a single server retrial queue with multiple vacations and different arrival rates is analyzed under the condition of stability. Some system performance

TABLE 3. Arrival rate λ_2 verses mean orbit size L_Q. ($\gamma = 5$, $\lambda_1 = 1.1$).

λ_2	Exponential	Erlang-2	Hyper exponential
	L_Q	L_Q	L_Q
.1	0.0826	0.1007	0.1072
.2	0.1638	0.1986	0.2115
.3	0.2438	0.2938	0.3132
.4	0.3225	0.3865	0.4125
.5	0.3999	0.4768	0.5095
.6	0.4762	0.5650	0.6045
.7	0.5512	0.6510	0.6979
.8	0.6252	0.7353	0.7897
.9	0.6980	0.8179	0.8804
1.0	0.7697	0.8990	0.9703

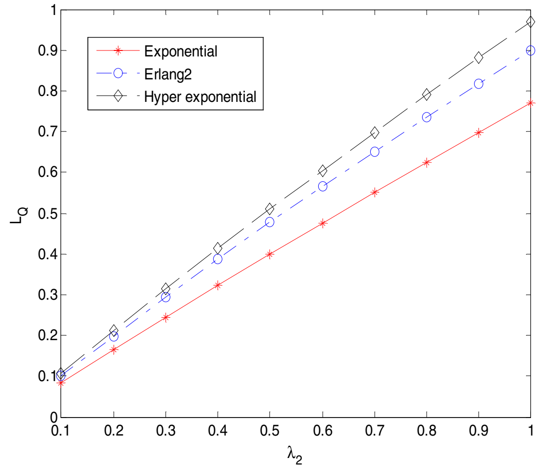

FIGURE 3. Arrival rate λ_2 verses mean orbit size L_Q.

measures, such as mean orbit size, mean system size, probability that the server is idle, probability that the server is busy and the probability that the server is on vacation are obtained. The effect of the parameters on the performance measures are illustrated graphically. Numerical illustrations are also presented.

APPENDIX A

The derivation of equations (3.18)–(3.21).

From equation (3.13) we have

$$P_0'(z) = \frac{\lambda_1}{\gamma} P_0(z) \frac{\left[\tilde{S}\left(\lambda_2 - \lambda_2 z\right) - 1\right]}{\left[z - \tilde{S}\left(\lambda_2 - \lambda_2 z\right)\right]} + \frac{1}{\gamma} \frac{\left[\tilde{V}\left(\lambda_2 - \lambda_2 z\right) - 1\right]}{\left[z - \tilde{S}\left(\lambda_2 - \lambda_2 z\right)\right]} \left[\sum_{l=1}^{\infty} V_{l,0}(0) + P_{1,0}(0)\right].$$

This is a linear type of equation. On solving this equation for $P_0(z)$ we have

$$P_0(z) = P_0(1)K(z) + K(z)\int_1^z \frac{g(t)}{K(t)\gamma\left[t - \tilde{S}\left(\lambda_2 - \lambda_2 t\right)\right]}\mathrm{d}t, \tag{A.1}$$

where

$$K(z) = \exp\left\{-\frac{\lambda_1}{\gamma}\int_1^z \left[\frac{\tilde{S}\left(\lambda_2 - \lambda_2 u\right) - 1}{u - \tilde{S}\left(\lambda_2 - \lambda_2 u\right)}\right]\mathrm{d}u\right\}.$$

$g(z) = [\tilde{V}(\lambda_2 - \lambda_2 z) - 1][\sum\limits_{l=1}^{\infty} V_{l,0}(0) + P_{l,0}(0)]$ which is equation (3.18).

From equation (3.14) we have

$$(\theta - \lambda_2 - \lambda_2 z)\,\tilde{P}_1(z,\theta) = P_1(z,0) - \gamma P_0'(z)\tilde{S}(\theta) - \lambda P_0(z)\tilde{S}(\theta).$$

Substituting $\theta = \lambda_2 - \lambda_2 z$ in the above equation we have

$$P_1(z,0) = \lambda_1 P_0(z)\tilde{S}\left(\lambda_2 - \lambda_2 z\right) + \gamma P_0'(z)\tilde{S}\left(\lambda_2 - \lambda_2 z\right). \tag{A.2}$$

Substituting for $P_1(z,0)$ from (A.2) into equation (3.14) we have

$$\tilde{P}_1(z,0) = \frac{\left[\tilde{S}\left(\lambda_2 - \lambda_2 z\right) - 1\right]\left[\lambda_1 P_0(z) + \gamma P_0'(z)\right]}{(-\lambda_2 + \lambda_2 z)} \tag{A.3}$$

which is equation (3.19).

From equation (3.15) we have

$$(\theta - \lambda_2 + \lambda_2 z)\,\tilde{V}_1(z,\theta) = V_1(z,0) - P_{1,0}(0)\tilde{V}(\theta).$$

Substituting $\theta = \lambda_2 - \lambda_2 z$ in the above equation we have

$$V_1(z,0) = P_{1,0}(0)\tilde{V}\left(\lambda_2 - \lambda_2 z\right). \tag{A.4}$$

Substituting for $V_1(z,0)$ from (A.4) into equation (3.15) we have

$$\tilde{V}_1(z,0) = \frac{\left[\tilde{V}\left(\lambda_2 - \lambda_2 z\right) - 1\right] P_{1,0}(0)}{(-\lambda_2 + \lambda_2 z)} \tag{A.5}$$

which is equation (3.20).

From equation (3.16) we have

$$(\theta - \lambda_2 + \lambda_2 z)\,\tilde{V}_l(z,\theta) = V_l(z,0) - V_{l-1,0}(0)\tilde{V}(\theta).$$

Substituting $\theta = \lambda_2 - \lambda_2 z$ in the above equation we have

$$V_l(z,0) = V_{l-1,0}(0)\tilde{V}\left(\lambda_2 - \lambda_2 z\right). \tag{A.6}$$

Substituting for $V_l(z,0)$ from (A.6) into equation (3.16) we have

$$\tilde{V}_l(z,0) = \frac{\left[\tilde{V}\left(\lambda_2 - \lambda_2 z\right) - 1\right] V_{l-1,0}(0)}{(-\lambda_2 + \lambda_2 z)} \tag{A.7}$$

which is equation (3.21).

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