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Cost Analysis of *MAP/G(a, b)/1/N* Queue with Multiple Vacations and Closedown Times

A. Senthil Vadivu[1,*] and R. Arumuganathan[2]

[1]Department of Mathematics, PSG Institute of Technology and Applied Research, Coimbatore, India
[2]Department of Mathematics, PSG College of Technology, Coimbatore, India
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Abstract: This paper gives the cost analysis of a finite capacity single server bulk queueing model with closedown times. The server serves the customers in batches of maximum size 'b' with a minimum threshold value 'a'. Customers arrive according to a Markovian Arrival Process (MAP). On completion of a service, if the queue length is less than 'a', then the server performs a closedown work and then leaves for a vacation of random length. When the server returns from vacation and if the queue length is still less than 'a' he avails another vacation and so on until the server finds 'a' customers waiting in the queue. After the completion of a service, if the number of customers in the queue is greater than a specified value 'a' then the server will continue the batch service with general bulk service rule. On the other hand, if the server finds at least 'a' customers during closedown period, he immediately starts serving the batch of 'a' customers. Using supplementary variable and imbedded Markov chain technique, queue length distribution at arbitrary epoch is obtained. Some key performance measures are also obtained. Cost model is discussed with Numerical illustration.

Keywords: Closedown times, cost analysis, MAP, multiple vacation.

1. Introduction

In recent years, increasing attention has been devoted to analyse the queueing systems with vacations using tractable point process called MAP as input. In general, MAP is a non renewal input process which includes the Markov modulated Poisson process (MMPP), the PH renewal process and the super positions of such process as particular cases. It was first introduced by Lucantoni *et al*. [17]. MAPs are used in traffic engineering to match correlated and/or bursty arrival processes commonly arising in computer and communication applications. For more details on these point process and their importance in stochastic modelling, one can refer to Neuts [20].

Vacation queue is an efficient and easy way to analyse the queues in cases where a single channel is serving more than one queue and are useful for the systems in which the server wants to utilise the idle time for different purposes. Application of server vacation models can be found in manufacturing systems, designing of local area networks and data communication systems. For more details one can refer to the comprehensive survey by Doshi [6]. Blondia [4] analysed *MAP/G/1/N* queue with multiple vacations for two types of service disciplines namely exhaustive service discipline and limited service discipline. Using embedded Markov chain and semigenerative techniques, Tadj *et al*. [27] analysed an optimal control of batch arrival, bulk service queueing system with *N*- policy with Bernoulli vacation schedule. Sikdar and Gupta [24], consider a finite-buffer batch arrival and batch service queue with single and multiple vacations.

* Corresponding author. E-mail: asenthilvadivu@gmail.com

Horvath and Telek [11], analysed canonical representation of phase type distributions using transformation method. Using matrix technique and method of catastrophes, Kim *et al*. [15] analysed the traffic control in telecommunications by considering generation of flows as MAP and service time follows phase type distribution. Telek and Horvath [28], investigate the problem of minimal representation of MAP(n) along with the discussion about characterization of phase type (PH) distributions. Blondia [3], analysed the single server queue with finite capacity where the arrival process is Neuts' versatile Markovian point process (the *N*-process).

Gupta and Sikdar [8], analysed a single server finite-buffer bulk-service queue in which the inter-arrival and service times are exponentially and arbitrarily distributed. Samanta *et al*. [22] analysed a discrete-time single-server finite-buffer batch arrival queue in which customers are served in batches according to a general bulk-service rule. Sikdar and Gupta [9, 23], analysed finite buffer batch service queueing system wherein the input process is MAP and for computational procedure service and vacation times follows phase type distribution.

In real time situations, most of the queueing situations occurring in communication networks and manufacturing systems are more complicated and need closedown time for further investigation. For example, in a pump manufacturing industry, in order to do other works such as making the templates for copy turning, checking the components etc., the operator always shuts down the machine and removes the templates before taking up other works. This motivates to consider closedown times.

The research on queueing systems with closedown time has been attempted by very few researchers. Arumuganathan and Jeyakumar [1-2] analysed some bulk queues with multiple vacations and closedown times. Jeyakumar and Arumganathan [12] investigated the non Markovian bulk queueing system with multiple vacations and controlled optional re-service with cost model. For more details on these bulk queues, reference can be made to Chaudhry and Templeton [5]. Ke [13] analysed, *M/G/1* system under NT policies with breakdowns, startup and closedown times. One can find the more general study of *MAP/G/1/N* queue with single and multiple vacations along with closedown time in Niu and Takahashi [21]. But cost analysis of the model is not considered for finite queues with bulk service. Sikdar [25] analysed, *MAP/G(a, b)/1/N* queue with multiple vacations, but without closedown time. A combination of *MAP/G(a, b)/1/N* queue with closedown time is worth investigating as it exists in many practical situations.

In this paper we consider finite queues with batch service, multiple vacations and closedown times in which the arriving customers are served by a single server in batches of maximum size 'b' with the minimum threshold value 'a'. On completion of a service, if the queue length is less than 'a', then the server performs a closedown work and then leaves for a vacation of random length. When the server returns from vacation and if the queue length is still less than 'a' he avails another vacation and so on until the server finds 'a' customers waiting in the queue. Furthermore, if the server finds 'a' customers during closedown period, he immediately starts serving the batch of 'a' customers. After the completion of a service, if the number of customers in the queue is greater than a specified value 'a' then the server will continue the batch service with general bulk service rule. The main motivation of this work comes from a real life situation of SVC based IP-over ATM networks which is based on the work of Jau-Chuan Ke [14]. In the IP over ATM networks, we can see a more complicated queueing situation where the close-down times are further needed. The close-down time here corresponds to an inactivity timer in the switched virtual channel connections (SVC) environment (see Hassan and Atiquzzaman [10]).

SVC is the abbreviation of switched virtual connection. The arrival of packets in SVC based IP-over ATM networks are correlated. The arriving packets are served by a single server namely SVC manager or IP controller with the general bulk service rule. After completion of service, system is having less than 'a' packets, server performs a closedown work such as starting an inactive timer of the SVC (e.g. routing information and bandwidth) which is reserved to anticipate more packets from the same IP flow. Furthermore, if 'a' packets arrive during closedown period, the server immediately serves minimum of 'a' packets in the system. But if less than 'a' packets are in the system after closedown time period, the server is doing another work (vacation) such as releasing the SVC by signalling protocols etc., when the server returns from vacation and the number of packets is still less than 'a' he avails another vacation and so on until the server finds 'a' packets waiting in the queue. This situation is modelled as *MAP/G(a, b)/1/N* queue with multiple vacations and closedown times.

The structure of the paper is organised as follows: Section 2 gives the description of the model and the notations used to describe the model parameters. In Section 3, using supplementary variable technique and embedded Markov chain technique, queue length distribution at various epochs such as service completion, closedown completion and vacation termination epochs is presented. In Section 4, relation between queue length distribution at arbitrary and various epochs are discussed. Some of the performance measures are obtained in Section 5. In Section 6, cost analysis of the model is obtained. Computational aspects, numerical illustration and conclusion are presented in Sections 7 and 8 respectively.

2. Mathematical Model

In this section, model description and the notations used to describe the model parameters are discussed.

2.1. Model Description

In this paper, a finite capacity single server bulk queueing system with multiple vacations and closedown time is considered. Arrival of customers is considered as a tractable Markovian arrival process. The service is done according to general bulk service rule. On completion of a service, if the queue length is less than 'a', then the server performs a closedown work and then leaves for a vacation of random length. When the server returns from vacation and if the queue length is still less than 'a' he avails another vacation and so on until the server finds 'a' customers waiting in the queue. If the number of customers in the queue is greater than a specified value 'a' then the server will continue the batch service with general bulk service rule. On the other hand, if the server finds at least 'a' customers during closedown period, he immediately starts serving the batch of 'a' customers. The graph showing the sample path of the proposed queueing model is depicted in Figure 1.

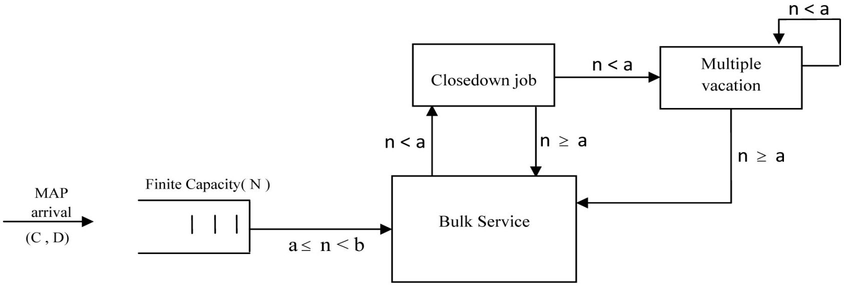

Figure 1. Schematic diagram.

2.2. Notations

The following notations are used for further development.

$S(x)[s(x)]\{S^*(\theta)\}$	DF[PDF]{LST} of the service time S of the typical Batch.
$V(x)[v(x)]\{V^*(\theta)\}$	DF[PDF]{LST} of a typical vacation time V of the server.
$U(x)[u(x)]\{U^*(\theta)\}$	DF[PDF]{LST} of a close down time of a server.
$\theta_s[\theta_u]\{\theta_v\}$	Mean service [close down] {vacation} time of a server.
ρ_1'	Probability that the server is busy.
$N_q(t)$	Number of customers present in the queue not counting those are in service at time t.
$J(t)$	State of the underlying Markov Chain of the MAP at time t.
$\hat{S}(t)$	Remaining service time of the batch in service at time t.
$\hat{V}(t)$	Remaining vacation time of the server at time t.
$\hat{U}(t)$	Remaining closedown time of the server at time t.

The state of the system at time 't' is described by the following random variables namely,

$$\xi(t)=\begin{cases}0, & \text{if the server is on vacation at time } t.\\ 1, & \text{if the server is busy at time } t.\\ 2, & \text{if the server is doing closedown job at time } t.\end{cases}$$

The service, closedown, vacation times are assumed to be i.i.d random variables and each is independent of the arrival process. The traffic intensity is given by $\rho=\lambda^*\theta_s/b$. It may be noted that in case of finite buffer queues ρ and ρ_1' are different. Also it is different in the case of infinite buffer queue with *GBS (a, b)* rule.

2.3. Markovian Arrival Process

The Markovian arrival process was introduced by Neuts and Lucantoni [19] as the versatile Markovian Process and later redefined as the MAP by Lucantoni *et al*. [17]. MAP is a generalization of the Poisson process, where the arrivals are governed by an underlying m–state Markov Chain. With probability $\mathbf{C}_{ij}, 1\le i, j\le m$, there is a transition from state i to state j without an arrival, and with probability $\mathbf{D}_{ij}, 1\le i, j\le m$, there is a transition from state i to state j with an arrival. The matrix $\mathbf{C}=[\mathbf{C}_{ij}]$ has non-negative off-diagonal and negative diagonal elements.

Let $A(t)$ denote the number of customers arriving in $(0, t]$ and $J(t)$ be the state of the underlying Markov Chain at time t with state space $(i:1\le i\le m)$. Then $\{A(t), J(t)\}$ is a two–dimensional Markov process with state space $\{(n, i): n\ge 0, 1\le i\le m\}$. The infinitesimal generator of the above Markov process is given by

$$Q=\begin{pmatrix}\mathbf{C} & \mathbf{D} & \mathbf{0} & \mathbf{0} & \cdot & \cdot & \cdot\\ \mathbf{0} & \mathbf{C} & \mathbf{D} & \mathbf{0} & \cdot & \cdot & \cdot\\ \mathbf{0} & \mathbf{0} & \mathbf{C} & \mathbf{D} & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \end{pmatrix}.$$

$\{A(t), J(t)\}$ is called the Markovian arrival process (MAP). Since Q is the infinitesimal generator of the MAP, we have $(\mathbf{C}+\mathbf{D})\mathbf{e}=0$. Where $\mathbf{e}$ is an $m\times 1$ vector with all its elements equal to 1. Since $(\mathbf{C}+\mathbf{D})$ is the infinitesimal generator of the underlying Markov Chain $\{J(t)\}$, there exists a stationary probability vector $\boldsymbol{\pi}$ such that $\boldsymbol{\pi}(\mathbf{C}+\mathbf{D})=0$, $\boldsymbol{\pi}\mathbf{e}=1$.

The fundamental arrival rate of the above Markov process is given by $\lambda^{*}=\boldsymbol{\pi}\,\mathbf{De}$.

2.4. Joint Probability Densities of the Model

The joint probability densities of queue length $N_q(t)$, state of the server $\xi(t)$ and the remaining service [closedown] {vacation} times are $\hat{S}(t)[\hat{U}(t)]\{\hat{V}(t)\}$ respectively, for $1\le i\le m$ is defined by

$$\pi_i(n,\, x:t)\Delta x = p\{N_q(t)=n,\, J(t)=i,\, x\le \hat{S}(t)\le x+\Delta x,\, \xi(t)=1\},\, 0\le n\le N,\, x\ge 0,$$

$$\eta_i(n,\, x:t)\Delta x = p\{N_q(t)=n,\, J(t)=i,\, x\le \hat{U}(t)\le x+\Delta x,\, \xi(t)=2\},\, 0\le n\le N,\, x\ge 0,$$

$$\omega_i(n,\, x:t)\Delta x = p\{N_q(t)=n,\, J(t)=i,\, x\le \hat{V}(t)\le x+\Delta x,\, \xi(t)=0\},\, 0\le n\le N,\, x\ge 0.$$

In steady state, that is when $t\to\infty$ the above probabilities will be denoted by

$$\lim_{t\to\infty}\pi_i(n,\, x:t)=\pi_i(n,\, x),\, x\ge 0,\, 0\le n\le N,$$

$$\lim_{t\to\infty}\eta_i(n,\, x:t)=\eta_i(n,\, x),\, x\ge 0,\, 0\le n\le N,$$

$$\lim_{t\to\infty}\omega_i(n,\, x:t)=\omega_i(n,\, x),\, x\ge 0,\, 0\le n\le N,$$

where $\pi_i(n, x)$ gives the density function with 'n' number of customers in the queue , phase of the arrival process is in state 'i' and remaining service time of the server is 'x'. Similarly other two terms can be interpreted.

Further, let us define the row vectors of the order $1\times m$ by

$$\boldsymbol{\pi}(n, x)=[\pi_1(n, x),\, \pi_2(n, x),\, \ldots,\, \pi_m(n, x)],$$

$$\boldsymbol{\eta}(n, x)=[\eta_1(n, x),\, \eta_2(n, x),\ldots,\, \eta_m(n, x)],$$

$$\boldsymbol{\omega}(n, x)=[\omega_1(n, x),\, \omega_2(n, x),\ldots,\, \omega_m(n, x)].$$

3. Queue Length Distribution at Various Epochs

This section gives the queue length distributions at arbitrary and various epochs such as service completion, closedown completion and vacation termination epochs.

3.1. Queue Length Distribution at Service Completion, Closedown Completion and Vacation Termination Epochs

It is assumed that either the service completion or closedown completion or vacation termination occurs at the time epochs t_0, t_1, t_2.......... Also the state of the system at t_i is defined as $\{N_q(t_i), \xi(t_i), J(t_i)\}$ where, $N_q(t_i), \xi(t_i)$ and $J(t_i)$ are as defined in Section 2. Therefore,

$$\xi(t_i)=\begin{cases}0, \text{ if the embedded point is vacation termination instant at time } t_i,\\ 1, \text{ if the embedded point is service completion instant at time } t_i,\\ 2, \text{ if the embedded point is closedown completion instant at time } t_i.\end{cases}$$

The limiting case of these probabilities is

$$\pi_j^+(n) = \lim_{i\to\infty} p(N_q(t_i) = n,\ J(t_i) = j,\ \xi(t_i) = 1),\ 0 \le n \le N,$$

$$\eta_j^+(n) = \lim_{i\to\infty} p(N_q(t_i) = n,\ J(t_i) = j,\ \xi(t_i) = 2),\ 0 \le n \le N,$$

$$\omega_j^+(n) = \lim_{i\to\infty} p(N_q(t_i) = n,\ J(t_i) = j,\ \xi(t_i) = 0),\ 0 \le n \le N.$$

The row vectors of the matrix are given by

$$\boldsymbol{\pi}^+(n) = [\pi_1^+(n), \pi_2^+(n)......................\pi_m^+(n)],$$

$$\boldsymbol{\eta}^+(n) = [\eta_1^+(n), \eta_2^+(n)......................\eta_m^+(n)],$$

$$\boldsymbol{\omega}^+(n) = [\omega_1^+(n), \omega_2^+(n)......................\omega_m^+(n)].$$

Further, let $\mathbf{A}(n)(\mathbf{C}(n))[\mathbf{V}(n)]$, $n \ge 0$ be the $m \times m$ matrices whose (i, j) th element represents the conditional probability that 'n' customers have been accepted during a service(closedown) [vacation] time of a batch and the underlying Markov chain is in phase 'j' at the end of the service(closedown) [vacation] time given that the underlying Markov chain was in phase 'i' at the beginning of the service(closedown) [vacation]. Further, let us denote $\mathbf{A}^c(n) = \sum_{k=n}^N \mathbf{A}(k)$, $\mathbf{V}^c(n) = \sum_{k=n}^N \mathrm{V}(k)$, $\mathbf{C}^c(n) = \sum_{k=n}^N \mathbf{C}(k)$, $1 \le n \le N$.

Observing the system immediately after an imbedded point, we have the Transition Probability Matrix (TPM) P with 9 block matrices of the form

$$P = \begin{pmatrix} \mathbf{SS} & \mathbf{SC} & \mathbf{0} \\ \mathbf{CS} & \mathbf{0} & \mathbf{CV} \\ \mathbf{VS} & \mathbf{0} & \mathbf{VV} \end{pmatrix}_{3(N+1)m \times 3(N+1)m}.$$

The first block **SS** gives the probability of transitions among the service completion epochs. The elements of these block is of the form

$$\mathbf{SS}_{(i,1)(j,1)} = \begin{cases} \mathbf{0},\ 0 \le i \le a-1, 0 \le j \le N, \\ \mathbf{A}(j),\ a \le i \le b, 0 \le j \le N-1, \\ \mathbf{A}(j-(i-b)),\ b+1 \le i \le N, 0 \le j \le N-1,\ i \ge b, \\ \mathbf{A}^c(j),\ a \le i \le b,\ j = N, \\ \mathbf{A}^c(j-(i-b)),\ b+1 \le i \le N,\ j = N,\ j \ge j-b, \\ \mathbf{0}, \text{otherwise.} \end{cases}$$

One of the other block **SC** of the TPM gives the probability of transitions from any service completion epoch to the closedown completion epochs. The structure of this block is given by

$$\mathbf{SC}_{(i,1)(j,2)} = \begin{cases} \mathbf{C}(j-i),\ 0 \le i \le a-1, 0 \le j \le N-1, \\ \mathbf{C}^c(j-i),\ 0 \le i \le a-1,\ j = N,\ j \ge i, \\ \mathbf{0}, \text{otherwise.} \end{cases}$$

One of the other block **CV** of the TPM gives the probability of transitions from any closedown completion to vacation termination epochs. The structure of this block is given by

$$\mathbf{CV}_{(i,\,2)(j,\,0)} = \begin{cases} \mathbf{V}(j-i), 0 \le i \le a-1, 0 \le j \le N-1,\ j \ge i, \\ \mathbf{V}^{c}(j-i), 0 \le i \le a-1, j = N,\ j \ge i, \\ \mathbf{0}, \text{ otherwise.} \end{cases}$$

The block **CS** of the TPM describes the probability of transition from every closedown completion to the next service completion epoch. Clearly this will be the service termination of the first batch after returning from closedown. The block **VS** of the TPM describes the probability of transition from every vacation termination epoch to the next service completion epochs respectively. Clearly this will be the service termination of the first batch after returning from vacation. For this reason the entire elements of the blocks **CS** and **VS** will be the same as the block **SS**. The last block **VV** of the TPM gives the probability of transitions among vacation termination epochs. The elements of the block will be same as **CV**.

Now we can obtain the unknown probabilities $\boldsymbol{\pi}^{+}(n)(0 \le n \le N)$, $\boldsymbol{\eta}^{+}(n)(0 \le n \le N)$, $\boldsymbol{\omega}^{+}(n)(0 \le n \le N)$ by solving the system of equations $[\boldsymbol{\pi}^{+}(n)\,\boldsymbol{\eta}^{+}(n)\,\boldsymbol{\omega}^{+}(n)] = [\boldsymbol{\pi}^{+}(n)\,\boldsymbol{\eta}^{+}(n)\,\boldsymbol{\omega}^{+}(n)]P$. This system of equations can be solved using GTH algorithm given in Latouche and Ramaswami [16].

3.2. Queue Length Distribution at Arbitrary Epoch

To determine queue length distributions at arbitrary epoch, we will develop relations between distributions of number of customers in the queue at service completion (vacation termination) [closedown] and arbitrary epochs. In order to apply supplementary variable method, we relate the states of the system at two consecutive time epochs t and $t+\Delta t$ and using probabilistic arguments, we have a set of partial differential equations for each phase $i, 1 \le i \le m$.

$$-\frac{d}{dx}\boldsymbol{\pi}(0,x) = \boldsymbol{\pi}(0,x)\mathbf{C} + s(x)\sum_{n=a}^{b}\left(\boldsymbol{\pi}(n,0)+\boldsymbol{\eta}(n,0)+\boldsymbol{\omega}(n,0)\right), \tag{1}$$

$$-\frac{d}{dx}\boldsymbol{\pi}(n,x) = \boldsymbol{\pi}(n,x)\mathbf{C} + \boldsymbol{\pi}(n-1,x)\mathbf{D} + s(x)\sum_{n=a}^{b}\left(\boldsymbol{\pi}(n+b,0)+\boldsymbol{\eta}(n+b,0)+\boldsymbol{\omega}(n+b,0)\right), \quad 1 \le n \le N-b, \tag{2}$$

$$-\frac{d}{dx}\boldsymbol{\pi}(n,x) = \boldsymbol{\pi}(n,x)\mathbf{C} + \boldsymbol{\pi}(n-1,x)\mathbf{D},\ N-b+1 \le n \le N-1, \tag{3}$$

$$-\frac{d}{dx}\boldsymbol{\pi}(N,x) = \boldsymbol{\pi}(n-1,x)\mathbf{D} + \boldsymbol{\pi}(N,x)\left(\mathbf{C}+\mathbf{D}\right), \tag{4}$$

$$-\frac{d}{dx}\boldsymbol{\eta}(0,x) = \boldsymbol{\eta}(0,x)\mathbf{C} + \boldsymbol{\pi}(0,0)u(x), \tag{5}$$

$$-\frac{d}{dx}\boldsymbol{\eta}(n,x) = \boldsymbol{\eta}(n,x)\mathbf{C} + \boldsymbol{\eta}(n-1,x)\mathbf{D} + u(x)\boldsymbol{\pi}(n,0),\ 1 \le n \le a-1, \tag{6}$$

$$-\frac{d}{dx}\boldsymbol{\eta}(n,x) = \boldsymbol{\eta}(n,x)\mathbf{C} + \boldsymbol{\eta}(n-1,x)\mathbf{D},\ \mathrm{a} \le n \le N-1, \tag{7}$$

$$-\frac{d}{dx}\boldsymbol{\eta}(N,x) = \boldsymbol{\eta}(N-1,x)\mathbf{D} + \boldsymbol{\eta}(N,x)\left(\mathbf{C}+\mathbf{D}\right), \tag{8}$$

$$-\frac{d}{dx}\boldsymbol{\omega}(0,x) = \boldsymbol{\omega}(0,x)\mathbf{C} + v(x)\left(\boldsymbol{\eta}(0,0)+\boldsymbol{\omega}(0,0)\right), \tag{9}$$

$$-\frac{d}{dx}\boldsymbol{\omega}(n,x) = \boldsymbol{\omega}(0,x)\mathbf{C} + \boldsymbol{\omega}(n-1,x)\mathbf{D} + v(x)\left(\boldsymbol{\eta}(n,0)+\boldsymbol{\omega}(n,0)\right),\ 1 \le n \le a-1, \tag{10}$$

$$-\frac{d}{dx}\boldsymbol{\omega}(n,x)=\boldsymbol{\omega}(n,x)\mathbf{C}+\boldsymbol{\omega}(n-1,x)\mathbf{D},\ a\le n\le N-1, \tag{11}$$

$$-\frac{d}{dx}\boldsymbol{\omega}(N,x)=\boldsymbol{\omega}(N,x)\big(\mathbf{C}+\mathbf{D}\big)+\boldsymbol{\omega}(N-1,x)\mathbf{D}. \tag{12}$$

The Laplace transforms of $\boldsymbol{\pi}(n,x)$, $\boldsymbol{\eta}(n,x)$ and $\boldsymbol{\omega}(n,x)$ are given by

$$\boldsymbol{\pi}^*(n,s)=\int_0^\infty e^{-sx}\boldsymbol{\pi}(n,x)dx,\ 0\le n\le N,\ \mathrm{Re}\, s\ge 0,$$

$$\boldsymbol{\eta}^*(n,s)=\int_0^\infty e^{-sx}\boldsymbol{\eta}(n,x)dx,\ 0\le n\le N,\ \mathrm{Re}\, s\ge 0,$$

$$\boldsymbol{\omega}^*(n,s)=\int_0^\infty e^{-sx}\boldsymbol{\omega}(n,x)dx,\ 0\le n\le N,\ \mathrm{Re}\, s\ge 0.$$

We have, $\boldsymbol{\pi}(n)=\boldsymbol{\pi}^*(n,0)=\int_0^\infty \boldsymbol{\pi}(n,x)dx,\ 0\le n\le N,$ also,

$$\boldsymbol{\eta}(n)=\boldsymbol{\eta}^*(n,0)=\int_0^\infty \boldsymbol{\eta}(n,x)dx,\ 0\le n\le N,$$

$$\boldsymbol{\omega}(n)=\boldsymbol{\omega}^*(n,0)=\int_0^\infty \boldsymbol{\omega}(n,x)dx,\ 0\le n\le N. \tag{13}$$

Multiplying Equations (1) to (12) by e^{-sx} and integrating with respect to x over 0 to ∞, we obtain the following transform equations:

$$-s\boldsymbol{\pi}^*(0,s)+\boldsymbol{\pi}(0,0)=\boldsymbol{\pi}^*(0,s)\mathbf{C}+\mathbf{S}^*(s)\sum_{n=a}^{b}\big(\boldsymbol{\pi}(n,0)+\boldsymbol{\eta}(n,0)+\boldsymbol{\omega}(n,0)\big), \tag{14}$$

$$-s\boldsymbol{\pi}^*(n,s)+\boldsymbol{\pi}(n,0)=\boldsymbol{\pi}^*(n,s)\mathbf{C}+\boldsymbol{\pi}^*(n-1,s)\mathbf{D}+\mathbf{S}^*(s)\big(\boldsymbol{\pi}(n+b,0)+\boldsymbol{\eta}(n+b,0)+\boldsymbol{\omega}(n+b,0)\big),\quad 1\le n\le N-b, \tag{15}$$

$$-s\boldsymbol{\pi}^*(n,s)+\boldsymbol{\pi}(n,0)=\boldsymbol{\pi}^*(n,s)\mathbf{C}+\boldsymbol{\pi}^*(n-1,s)\mathbf{D},\ N-b+1\le n\le N-1, \tag{16}$$

$$-s\boldsymbol{\pi}^*(N,s)+\boldsymbol{\pi}(N,0)=\boldsymbol{\pi}^*(N,s)\big(\mathbf{C}+\mathbf{D}\big)+\boldsymbol{\pi}^*(N-1,s)\mathbf{D}, \tag{17}$$

$$-s\boldsymbol{\eta}^*(0,s)+\boldsymbol{\eta}(0,0)=\boldsymbol{\eta}^*(0,s)\mathbf{C}+\mathbf{U}^*(s)\boldsymbol{\pi}(0,0), \tag{18}$$

$$-s\boldsymbol{\eta}^*(n,s)+\boldsymbol{\eta}(n,0)=\boldsymbol{\eta}^*(n,s)\mathbf{C}+\boldsymbol{\eta}^*(n-1,s)\mathbf{D}+\mathbf{U}^*(s)\boldsymbol{\pi}(n,0),\ 1\le n\le a-1, \tag{19}$$

$$-s\boldsymbol{\eta}^*(n,s)+\boldsymbol{\eta}(n,0)=\boldsymbol{\eta}^*(n,s)\mathbf{C}+\boldsymbol{\eta}^*(n-1,s)\mathbf{D},\ a\le n\le N-1, \tag{20}$$

$$-s\boldsymbol{\eta}^*(N,s)+\boldsymbol{\eta}(N,0)=\boldsymbol{\eta}^*(N,s)\big(\mathbf{C}+\mathbf{D}\big)+\boldsymbol{\eta}^*(N-1,s)\mathbf{D}, \tag{21}$$

$$-s\boldsymbol{\omega}^*(0,s)+\boldsymbol{\omega}(0,0)=\boldsymbol{\omega}^*(0,s)\mathbf{C}+\mathbf{V}^*(s)\big(\boldsymbol{\omega}(0,0)+\boldsymbol{\eta}(0,0)\big), \tag{22}$$

$$-s\boldsymbol{\omega}^*(n,s)+\boldsymbol{\omega}(n,0)=\boldsymbol{\omega}^*(n,s)\mathbf{C}+\boldsymbol{\omega}^*(n-1,s)\mathbf{D}+\mathbf{V}^*(s)\big(\boldsymbol{\omega}(n,0)+\boldsymbol{\eta}(n,0)\big),\ 1\le n\le a-1, \tag{23}$$

$$-s\boldsymbol{\omega}^*(n,s)+\boldsymbol{\omega}(n,0)=\boldsymbol{\omega}^*(n,s)\mathbf{C}+\boldsymbol{\omega}^*(n-1,s)\mathbf{D},\ a\le n\le N-1, \tag{24}$$

$$-s\boldsymbol{\omega}^*(N,s)+\boldsymbol{\omega}(N,0)=\boldsymbol{\omega}^*(N,s)\big(\mathbf{C}+\mathbf{D}\big)+\boldsymbol{\omega}^*(N-1,s)\mathbf{D}, \tag{25}$$

Using the Equations (14) to (25), we derive certain results in the form of lemmas.

Lemma 1

The entering rate to the closedown or vacation states are equal to the departure rate from the closedown state or vacation state in an arbitrary slot, which is given by

$$\sum_{n=0}^{a-1} \boldsymbol{\pi}(n,0)\mathbf{e} = \sum_{n=a}^{N} \left(\boldsymbol{\eta}(n,0) + \boldsymbol{\omega}(n,0)\right) \mathbf{e}.$$

Proof is in the Appendix.

Lemma 2

The entering rate to closedown state is equal to the departure rate from the closedown state in an arbitrary slot, which is given by

$$\sum_{n=0}^{a-1} \boldsymbol{\pi}(n,0)\mathbf{e} = \sum_{n=0}^{N} \boldsymbol{\eta}(n,0)\mathbf{e}.$$

Proof is in the Appendix.

Lemma 3

The entering rate to vacation state is equal to the departure rate from the vacation state in an arbitrary slot, which is given by

$$\sum_{n=0}^{a-1} \left(\boldsymbol{\eta}(n,0)\right)\mathbf{e} = \sum_{n=a}^{n} \boldsymbol{\omega}(n,0)\mathbf{e}.$$

Proof is in the Appendix.

Lemma 4

1. Mean number of customers served per slot $[\sum_{n=0}^{N} \boldsymbol{\pi}(n,0)\mathbf{e}]$, multiplied by the mean service time of a server θ_s , gives the probability that the server is busy ρ_1' . That is

$$\sum_{n=0}^{N} \boldsymbol{\pi}(n)\mathbf{e} = \theta_S \left[\sum_{n=0}^{N} \boldsymbol{\pi}(n,0)\mathbf{e} \right] = \rho_1'.$$

2. Mean number of closedowns terminated per slot multiplied by the mean closedown time of a server gives the probability that the server is in closedown job, which is given by

$$\sum_{n=0}^{N} \boldsymbol{\eta}(n)\mathbf{e} = \theta_u \left[\sum_{n=0}^{N} \boldsymbol{\eta}(n,0)\mathbf{e} \right] = 1 - \rho_1' - \rho_2'.$$

3. Mean number of vacations terminated per slot multiplied by the mean vacation time of a server gives the probability that the server is on vacation, which is given by

$$\sum_{n=0}^{N} \boldsymbol{\omega}(n)\mathbf{e} = \theta_v \left[\sum_{n=0}^{N} \boldsymbol{\omega}(n,0)\mathbf{e} \right] = \rho_2'.$$

Proof is in the Appendix.

4. Relation Between Queue Length Distribution at Arbitrary and Various Epochs

In this section relationship between the queue length distribution at arbitrary and various epochs such as service completion, closedown completion and vacation termination epochs and pre arrival epoch are discussed.

4.1. Relation Between Queue Length Distributions at Arbitrary and Service Completion (Closedown) [Vacation Termination] Epochs

The relationship between the service completion (close down) [vacation termination] probabilities $\boldsymbol{\pi}^+(n)(\boldsymbol{\eta}^+(n))[\boldsymbol{\omega}^+(n)]$ with $\boldsymbol{\pi}(n,0)(\boldsymbol{\eta}(n,0))[\boldsymbol{\omega}(n,0)]$ are given by

$$\boldsymbol{\pi}^{+}(n)=\frac{1}{\sigma}\boldsymbol{\pi}(n,0)\,,\quad \boldsymbol{\omega}^{+}(n)=\frac{1}{\sigma}\boldsymbol{\omega}\,(n,0)\quad\text{and}\quad \boldsymbol{\eta}^{+}(n)=\frac{1}{\sigma}\boldsymbol{\eta}(n,0),\ 0\le n\le N, \tag{26}$$

where $\sigma=\sum_{n=0}^{N}\left(\boldsymbol{\pi}(n,0)+\boldsymbol{\omega}\,(n,0)+\boldsymbol{\eta}\,(n,0)\right)\mathbf{e}$.

Theorem 1

The arbitrary epoch probabilities for service completion, closedown and vacation termination epochs are given by

$$\boldsymbol{\pi}(0)=\sigma\left[\sum_{n=a}^{b}(\boldsymbol{\pi}^{+}(n)+\boldsymbol{\eta}^{+}(n)+\boldsymbol{\omega}^{+}(n))-\boldsymbol{\pi}^{+}(0)\right](-\mathbf{C})^{-1},$$

$$\boldsymbol{\pi}(n)=(\boldsymbol{\pi}(n-1)\mathbf{D}+\sigma[(\boldsymbol{\pi}^{+}(n+b)+\boldsymbol{\eta}^{+}(n+b)+\boldsymbol{\omega}^{+}(n+b))-\boldsymbol{\pi}^{+}(n)])(-\mathbf{C})^{-1},\ 1\le n\le N-b,$$

$$\boldsymbol{\pi}(n)=[\boldsymbol{\pi}(n-1)\mathbf{D}-\sigma\boldsymbol{\pi}^{+}(n)](-\mathbf{C})^{-1},\ N-b+1\le n\le N-1,$$

$$\boldsymbol{\eta}(0)=\sigma[\boldsymbol{\pi}^{+}(0)-\boldsymbol{\eta}^{+}(0)](-\mathbf{C})^{-1},$$

$$\boldsymbol{\eta}(n)=[\boldsymbol{\eta}(n-1)\mathbf{D}+\sigma(\boldsymbol{\pi}^{+}(n)-\boldsymbol{\eta}^{+}(n))](\text{-}\mathbf{C})^{-1},\ 1\le n\le a-1,$$

$$\boldsymbol{\eta}(n)=[\boldsymbol{\eta}(n-1)\mathbf{D}-\sigma\boldsymbol{\eta}^{+}(n)](-\mathbf{C})^{-1},\ a\le n\le N-1,$$

$$\boldsymbol{\omega}(0)=\sigma[\boldsymbol{\eta}^{+}(0)](-\mathbf{C})^{-1},$$

$$\boldsymbol{\omega}(n)=[\sigma\boldsymbol{\eta}^{+}(n)+\boldsymbol{\omega}(n-1)\mathbf{D}](-\mathbf{C})^{-1},\ 1\le n\le a-1,$$

$$\boldsymbol{\omega}(n)=[\boldsymbol{\omega}(n-1)\mathbf{D}-\sigma\boldsymbol{\omega}^{+}(n)](-\mathbf{C})^{-1},\ a\le n\le N-1.$$

Proof: By making use of relations Equation (26), we will determine arbitrary epoch probabilities in terms of service completion or close down or vacation termination epoch probabilities. Multiplying Equations (a1) to (a3) by $1/\sigma$ and using Equation (26), we get

$$\boldsymbol{\pi}^{+}(0)=\frac{1}{\sigma}\boldsymbol{\pi}(0)\mathbf{C}+\sum_{n=a}^{b}\left(\boldsymbol{\pi}^{+}(n)+\boldsymbol{\eta}^{+}(n)+\boldsymbol{\omega}^{+}(n)\right), \tag{27}$$

$$\boldsymbol{\pi}^{+}(n)=\frac{1}{\sigma}\boldsymbol{\pi}(n)\mathbf{C}+\frac{1}{\sigma}\boldsymbol{\pi}(n-1)\mathbf{D}+\left(\boldsymbol{\pi}^{+}(n+b)+\boldsymbol{\eta}^{+}(n+b)+\boldsymbol{\omega}^{+}(n+b)\right), \tag{28}$$

$$\boldsymbol{\pi}^{+}(n)=\frac{1}{\sigma}\boldsymbol{\pi}(n)\mathbf{C}+\frac{1}{\sigma}\boldsymbol{\pi}(n-1)\mathbf{D}. \tag{29}$$

Simplifying the above three equations, we get

$$\boldsymbol{\pi}(0)=\sigma\left[\sum_{n=a}^{b}\left(\boldsymbol{\pi}^{+}(n)+\boldsymbol{\eta}^{+}(n)+\boldsymbol{\omega}^{+}(n)\right)-\boldsymbol{\pi}^{+}(0)\right]\left(-\mathbf{C}\right)^{-1}, \tag{30}$$

$$\boldsymbol{\pi}(n)=\left(\boldsymbol{\pi}(n-1)\mathbf{D}+\sigma\left[\left(\boldsymbol{\pi}^{+}(n+b)+\boldsymbol{\eta}^{+}(n+b)+\boldsymbol{\omega}^{+}(n+b)\right)-\boldsymbol{\pi}^{+}(n)\right]\right)\left(-\mathbf{C}\right)^{-1},\ 1\le n\le N-b, \tag{31}$$

$$\boldsymbol{\pi}(n)=\left[\boldsymbol{\pi}(n-1)\mathbf{D}-\sigma\ \boldsymbol{\pi}^{+}(n)\right]\left(-\mathbf{C}\right)^{-1},\ N-b+1\le n\le N-1. \tag{32}$$

Multiplying Equations (a5) to (a7) by $1/\sigma$ and using Equation (26) and simplifying, we get

$$\boldsymbol{\eta}(0)=\sigma\ \left[\boldsymbol{\pi}^{+}(0)-\boldsymbol{\eta}^{+}(0)\right]\left(-\mathbf{C}\right)^{-1}, \tag{33}$$

$$\boldsymbol{\eta}(n)=\left[\boldsymbol{\eta}(n-1)\mathbf{D}+\sigma\ \left(\boldsymbol{\pi}^{+}(n)-\boldsymbol{\eta}^{+}(n)\right)\right]\left(-\mathbf{C}\right)^{-1},\ 1\le n\le a-1, \tag{34}$$

$$\boldsymbol{\eta}(n)=\left[\boldsymbol{\eta}(n-1)\mathbf{D}-\sigma\ \boldsymbol{\eta}^{+}(n)\right]\left(-\mathbf{C}\right)^{-1},\ a\le n\le N-1. \tag{35}$$

Multiplying Equations (a9) to (a11) by $1/\sigma$ and using Equation (26) and simplifying, we get

$$\boldsymbol{\omega}(0) = \sigma\left[\ \boldsymbol{\eta}^{+}(0)\right](-\mathbf{C})^{-1},$$

$$\boldsymbol{\omega}(n) = \left[\sigma\boldsymbol{\eta}^{+}(n) + \boldsymbol{\omega}(n-1)\mathbf{D}\right](-\mathbf{C})^{-1},\ 1 \le n \le a-1,$$

$$\boldsymbol{\omega}(n) = \left[\boldsymbol{\omega}(n-1)\mathbf{D} - \sigma\ \boldsymbol{\omega}^{+}(n)\right](-\mathbf{C})^{-1},\ a \le n \le N-1.$$

It may be noted that we do not have such expression for $\boldsymbol{\pi}(N)$, $\boldsymbol{\eta}(N)$ and $\boldsymbol{\omega}(N)$. However, the values of $\boldsymbol{\pi}(N)\ \mathbf{e}$, $\boldsymbol{\eta}(N)\ \mathbf{e}$ and $\boldsymbol{\omega}(N)\ \mathbf{e}$ can be obtained by using the Lemma 4 and it is given by

$$\boldsymbol{\pi}(N)\ \mathbf{e} = \rho_1' - \sum_{n=0}^{N-1}\boldsymbol{\pi}(n)\ \mathbf{e},\ \boldsymbol{\omega}(N)\ \mathbf{e} = \rho_2' - \sum_{n=0}^{N-1}\boldsymbol{\omega}(n)\ \mathbf{e}\ \text{ and }\ \boldsymbol{\eta}(N)\ \mathbf{e} = 1 - \rho_1' - \rho_2' - \sum_{n=0}^{N-1}\boldsymbol{\omega}(n)\ \mathbf{e}$$

respectively.

Though the vectors $\boldsymbol{\pi}(N)$, $\boldsymbol{\eta}(N)$ and $\boldsymbol{\omega}(N)$ are not found component wise, $\boldsymbol{\pi}(N)\mathbf{e}$, $\boldsymbol{\eta}(N)\ \mathbf{e}$ and $\boldsymbol{\omega}(N)\ \mathbf{e}$ are sufficient to determine key performance measures.

The unknown quantities ρ_1' and σ which are involved in the above expressions can be obtained with the help of the Theorems and lemmas given below.

Theorem 2

The probability that the server is busy is given by

$$\rho_1' = \frac{\theta_s \sum_{n=0}^{N}\boldsymbol{\pi}^{+}(n)\ \mathbf{e}}{\theta_s \sum_{n=0}^{N}\boldsymbol{\pi}^{+}(n)\ \mathbf{e} + \theta_u \sum_{n=0}^{N}\boldsymbol{\eta}^{+}(n)\ \mathbf{e} + \theta_v \sum_{n=0}^{N}\boldsymbol{\omega}^{+}(n)\ \mathbf{e}}.$$

Proof: Let $\Theta_b(\Theta_i)\{\Theta_c\}$ be the random variables denoting the length of busy (vacation) {close down} period and $\theta_b(\theta_i)\{\theta_c\}$ be the mean length of busy (vacation) {close down} period, then we have

$$\rho_1' = \frac{\theta_b}{\theta_b + \theta_i + \theta_c} = \frac{\sum_{n=0}^{N}\boldsymbol{\pi}(n)\mathbf{e}}{\sum_{n=0}^{N}\boldsymbol{\pi}(n)\mathbf{e} + \sum_{n=0}^{N}\boldsymbol{\eta}(n)\mathbf{e} + \sum_{n=0}^{N}\boldsymbol{\omega}(n)\mathbf{e}}.$$

Using Lemma 4, dividing numerator and denominator by σ and using Equation (26), we get

$$\rho_1' = \frac{\theta_s \sum_{n=0}^{N}\boldsymbol{\pi}^{+}(n)\ \mathbf{e}}{\theta_s \sum_{n=0}^{N}\boldsymbol{\pi}^{+}(n)\ \mathbf{e} + \theta_u \sum_{n=0}^{N}\boldsymbol{\eta}^{+}(n)\ \mathbf{e} + \theta_v \sum_{n=0}^{N}\boldsymbol{\omega}^{+}(n)\ \mathbf{e}}.$$

Lemma 5

The expression for σ in terms of ρ_1' and ρ_2' is given by

$$\sigma = \frac{\theta_v\theta_u\rho_1' + \theta_s\theta_u\rho_2' + \theta_v\theta_s(1 - \rho_1' - \rho_2')}{\theta_v\theta_u\theta_s}.$$

Proof is in the Appendix.

Lemma 6

Let $\mathbf{p}(n),\ 0 \le n \le N$ be the $1 \times m$ vector whose jth component is the probability of n customers in the queue at arbitrary epoch and the state of the process is j. Then, in vector notation $\mathbf{p}(n)$ is given by

$$\mathbf{p}(n) = \boldsymbol{\omega}(n) + \boldsymbol{\pi}(n) + \boldsymbol{\eta}(n),\ 0 \le n \le N-1,$$

$$\mathbf{p}(N) = \overline{\boldsymbol{\pi}} - \sum_{n=0}^{N-1} \left(\boldsymbol{\omega}(n) + \boldsymbol{\pi}(n) + \boldsymbol{\eta}(n) \right).$$

4.2. Queue Length Distribution at Pre-arrival Epoch

Let $\mathbf{p}^{-}(n)$ be the $1 \times m$ vector whose jth component is given by $\mathbf{p}_j^{-}(n)$ and it gives the steady state probability that an arrival finds n ($0 \le n \le N$) customers in the queue and the arrival process is in state j. Then the vector $\mathbf{p}^{-}(n)$ is given by $\mathbf{p}^{-}(n) = \mathbf{p}(n)\mathbf{D}/\lambda^{*},\ 0 \le n \le N$.

5. Performance Measures

As the steady state probabilities at service completion, close down, vacation termination, departure and arbitrary epochs are derived, various performance measures of the queue are obtained in this section

a. Average number of customers in the queue at an arbitrary epoch $(L_q = \sum_{n=0}^{N} n\,\mathbf{p}(n)\mathbf{e})$.

b. Average number of customers in the queue when the server is busy $(L_{q_1} = \sum_{n=0}^{N} n\boldsymbol{\pi}(n)\mathbf{e})$.

c. Average number in the queue when the server is on vacation $(L_{q_0} = \sum_{n=0}^{N} n\boldsymbol{\omega}(n)\mathbf{e})$.

d. Average number in the queue when the server is on closedown work $(L_{q_2} = \sum_{n=0}^{N} n\boldsymbol{\eta}(n)\mathbf{e})$.

e. The loss probability of an arrival customer is given by $P_{loss} = \mathbf{p}^{-}(N)\mathbf{e} = \mathbf{p}(N)\mathbf{De}/\lambda^{*}$.

6. Cost Analysis

Optimization of cost is necessary to control any system economically. In this section a cost model is designed and optimal thresholds is sought that yield minimum cost. Cost model is built according to the following considerations:

- Activating and deactivating the server result in fixed start-up and shut-down costs, respectively.
- When the server is turned off, an idle cost for power, heat, maintenance, etc. is charged, and when the server is turned on, attendant, fuel, or other costs may be added to the dormant cost to form the running cost.
- The holding cost is a penalty for delaying a customer in the system.
- When the server is taking vacations, reward due to vacations is considered.
- Due to space limitations (finite capacity system) fixed cost for each lost customer is considered.

The survey of Tadj and Choudhury [26], shows the importance of optimization techniques in queueing systems. Total expected cost function per unit time gives the optimal values of the queueing system. This section is devoted to find these measures.

6.1. Cost Model

Cost analysis is important for any managerial decision. In this section we derive the total average cost of the proposed queueing system. In Sections 6.1.1. and 6.1.2., we derive the expected busy period $E(B)$ and expected idle period $E(I)$ which are used in total average cost in 6.1.3.

6.1.1. Busy Period Analysis

Let B be the busy period random variable. Then we define the random variable J by

$$J = 0, \text{ if the server finds less than 'a' customers after the first service.}$$
$$= 1, \text{ if the server finds 'a' customers after the first service.}$$

Then the expected length of busy period is given by

$$E(B) = E(B/J=0)P(J=0) + E(B/J=1)P(J=1)$$
$$= E(S)P(J=0) + \{E(S)+E(B)\}P(J=1).$$

Then solving for $E(B)$, we get

$$E(B) = \frac{E(S)}{P(J=0)} = \frac{E(S)}{\sum_{n=0}^{a-1} \mathbf{p}^{+}(n)}, \quad \text{(Refer p.324, [5])}$$

where $\mathbf{p}^{+}(n)$ is the steady state probability vector that 'n' customers are left at a departure epoch of a batch, which is give by $\mathbf{p}^{+}(n) = \boldsymbol{\pi}^{+}(n)\mathbf{e} / \sum_{n=0}^{N} \boldsymbol{\pi}^{+}(n)\mathbf{e}, 0 \le n \le N$. (Refer Gupta and Sikdar [8]).

6.1.2. Idle Period Analysis

In this model, if the server finds less than ‘a’ customers in the system, then closedown work followed by a vacation of random length takes place. After returning from vacation, still if less than ‘a’ customers are found and another vacation is made and so on. So we need to calculate ‘Idle period due to multiple vacations’ and let that random variable be I_1. On the other hand when the server finds ‘a’ customers during closedown period, then immediately service is done to the batch of ‘a’ customers without taking vacation. Therefore we need to calculate ‘idle period due to close down’ and let it be I_2.

Let I be the random variable denoting idle period. Then the expected length of idle period is given by $E(I) = E(I_1) + E(I_2)$, which is obtained as follows:

Define the random variable U as

$$U = 0, \text{ if the server finds 'a' customers after the first closedown period.}$$
$$= 1, \text{ if the server finds less than 'a' customers after the first closedown period.}$$

Then the expected length of idle period due to closedown period is given by

$$E(I_2) = E(I_2/U=0)P(U=0) + E(I_2/U=1)P(U=1)$$
$$= E(C)P(U=0) + \{E(C)+E(I_2)\}P(U=1).$$

Then solving for $E(I_2)$, we get

$$E(I_2) = \frac{E(C)}{P(U=0)} = \frac{E(C)}{1 - \sum_{i=0}^{a-1} \left(\frac{\boldsymbol{\eta}^{+}(i)\mathbf{e}}{\sum_{n=0}^{N} \boldsymbol{\eta}^{+}(n)\mathbf{e}} \right)}.$$

Similarly, expected length of idle period due to multiple vacations is given by

$$E(I_1) = \frac{E(V)}{P(U=0)} = \frac{E(V)}{1 - \sum_{i=0}^{a-1} \left(\frac{\boldsymbol{\omega}^+(i)\mathbf{e}}{\sum_{n=0}^{N} \boldsymbol{\omega}^+(n)\mathbf{e}} \right)}.$$

6.1.3. Total Average Cost

To derive the Total average cost we use the following notations. Let C_s be the start-up cost, C_h be the holding cost per customer per unit time, C_0 be the operating cost per unit time, C_r be the reward cost per unit time due to vacation, C_1 be the fixed cost for each lost customer due to finite capacity.

The length of the cycle is the sum of the idle period and the busy period.

Therefore the expected length of cycle is given by $E(T_c) = E(I) + E(B)$.

Now the total average cost per unit time is given by

Total Average Cost = Start-up cost per unit time
+ holding cost of the number of customers in the queue per unit time
+ operating cost per unit time × ρ
– Reward due to vacation per unit time
+ fixed cost for each lost customer when the system is blocked due to space limitations.

Total Average Cost = $\left[C_s - C_r \times (E(V)/P(U=0))\right](1/E(T_c)) + C_h L_q + C_0 \rho + C_l\, P_{loss}$,

where $\rho = \lambda^* \theta_s / b$.

7. Computational Aspects

In this section, we discuss various steps needed for the computation of the matrices $\mathbf{A}(n), \mathbf{V}(n), \mathbf{C}(n)$, $n \geq 0$ of the TPM P. In General, the evaluation of the matrices $\mathbf{A}(n)(\mathbf{C}(n))[\mathbf{V}(n)]$ for arbitrary service (closedown)[vacation] time distribution requires numerical integration and can be carried out along the lines proposed by Lucantoni and Ramaswami [18]. According to Neuts [20], when the service distribution is of phase type, these matrices can be evaluated without any numerical integration. Also, PH distribution is a rich class of distribution, service (vacation) [closedown] time distributions arising in the real world queueing problems that can be easily approximated by it. For computation purpose, let us assume that $S(x)$ follows a PH-distribution with irreducible representation $(\boldsymbol{\beta}, \mathbf{S})$, where $\boldsymbol{\beta}$ and $\mathbf{S}$ are of dimensions m_1. Similarly let $V(x)$ and $C(x)$ follows a PH-distribution with irreducible representations $(\boldsymbol{\alpha}, \mathbf{T})$, $(\boldsymbol{\gamma}, \mathbf{U})$, where $\boldsymbol{\alpha}$ and $\mathbf{T}$, $\boldsymbol{\gamma}$ and $\mathbf{U}$ are of dimensions m_2, m_3 respectively. Then the matrices $\mathbf{A}$(n), $\mathbf{V}(n)$ and $\mathbf{C}(n)$ can be computed using the procedure described in Neuts [20], Gupta and Laxmi [7].

7.1. Numerical Illustration

The cost analysis of MAP/ $E_2^{1,10}$ /1/50 queue with multiple vacation and closedown times (both follow PH – distribution) are given in the following input parameters:

$$\mathbf{C} = \begin{pmatrix} -4.475 & 1.35 \\ 1.695 & -3.275 \end{pmatrix}; \quad \mathbf{D} = \begin{pmatrix} 1.475 & 1.65 \\ 0.305 & 1.275 \end{pmatrix},$$

$$\boldsymbol{\alpha} = (0.7 \quad 0.3); \quad \mathbf{T} = \begin{pmatrix} -1.088 & 0.068 \\ 0.061 & -1.432 \end{pmatrix}; \quad \theta_v = 0.8976,$$

$$\boldsymbol{\beta} = (1.0 \quad 0.0); \quad \mathbf{S} = \begin{pmatrix} -0.57143 & 0.57143 \\ 0.0 & -0.57143 \end{pmatrix}; \quad \theta_S = 3.5000,$$

$$\gamma = (0.4 \quad 0.6); \quad \mathbf{U} = \begin{pmatrix} -2.235 & 1.985 \\ 1.054 & -1.854 \end{pmatrix}; \quad \theta_c = 1.7105,$$

with $m = 2$, $\lambda^* = 2.1980$ and $\rho = 0.7693$.

Tables 1 and 2 give the performance measures, total average cost and the unknown probabilities for various numbers of customers in the system with and without closedown times. From these two tables, one can observe the following:

- Expected number of customers in the system and the server's busy period increases in without closedown times.
- Server's idle period decreases and the total average cost increases in without closedown time.

Table 1. Performance measures and unknown probabilities for various number of customers in the system when $N = 25$ (With closedown time).

a	E(Q)	E(B)	E(I)	Total average cost	$\pi^+(n)$		$\omega^+(n)$		$\eta^+(n)$	
					K=1	K=2	K=1	K=2	K=1	K=2
1	4.458652	0.688971	0.060553	11.246222	0.02652647	0.01754913	0.00003172	0.00001883	0.00001211	0.00000678
2	4.531694	0.643209	0.159838	10.381957	0.02648340	0.01752299	0.00022330	0.00013228	0.00008152	0.00004558
3	3.924525	0.539133	0.251845	9.573141	0.02680649	0.01783482	0.00143084	0.00084695	0.00062707	0.00035130
4	3.756351	0.469630	0.326363	8.955703	0.02484903	0.01659649	0.00480069	0.00284005	0.00220663	0.00123866
5	3.312172	0.395936	0.345982	8.603588	0.02248363	0.01509596	0.01401482	0.00830171	0.00686748	0.00387654
6	3.505264	0.371889	0.381384	8.422668	0.01984007	0.01333326	0.02463060	0.01474903	0.01128984	0.00647841
7	**3.481549**	**0.348867**	**0.377857**	**8.373304**	0.01672000	0.01226618	0.03777051	0.02594161	0.01639091	0.01100920
8	3.184589	0.327722	0.331598	8.435056	0.01495531	0.01100110	0.03243166	0.02229791	0.01447377	0.00974227
9	3.417872	0.300355	0.446723	8.590582	0.01355628	0.01001581	0.02738454	0.01886372	0.01284156	0.00867485
10	3.642044	0.314771	0.320034	8.679445	0.01289333	0.00951669	0.02854311	0.01961353	0.01235119	0.00833160

Table 2. Performance measures and unknown probabilities for various number of customers in the system when $N = 25$ (Without closedown time).

a	E(Q)	E(B)	E(I)	Total average cost	$\pi^+(n)$		$\omega^+(n)$	
					K=1	K=2	K=1	K=2
2	4.289986	0.673164	0.055695	10.864881	0.02495144	0.01650782	0.00029585	0.00017566
3	4.201676	0.657518	0.061612	10.592852	0.02284754	0.01511706	0.00072986	0.00043327
4	3.788409	0.599739	0.088202	10.339895	0.02204008	0.01458724	0.00221654	0.00131250
5	3.419378	0.537052	0.116147	9.6257039	0.02269932	0.01517061	0.01366927	0.00808629
6	3.540089	0.512297	0.145304	9.495535	0.01870086	0.01242118	0.01004058	0.00594061
7	3.662333	0.485285	0.170346	9.262158	0.01672000	0.01226618	0.03777051	0.02594161
8	**3.755745**	**0.456334**	**0.187217**	**8.657669**	0.01495531	0.01100110	0.03243166	0.02229791
9	3.777455	0.425396	0.191043	9.042243	0.01456480	0.00972436	0.03767304	0.02252267
10	4.395549	0.430757	0.202887	9.667246	0.01289333	0.00951669	0.02854311	0.01961353

Tables 3 and 4 give the unknown probabilities for bulk service with various number of customers in the system for the system capacity $N = 50$ and $N = 25$ respectively.

Table 5 gives the total average cost for various system capacities with the fixed threshold value. From this, total average cost decreases with increase in the system capacity for the threshold value $a = 7$. The cost values assumed in this model are $C_s = 4;\ C_h = 0.50;\ C_0 = 5;\ C_r = 1;\ \ C_u = 0.25; C_l = 6.$

Table 3. Unknown probabilities for bulk service vs. various number of customers in the system when $N = 50$.

n/a	1	2	3	4	5	6	7	8	9	10
0	0.064314	0.063509	0.068475	0.058482	0.054326	0.049212	0.043118	0.035833	0.026889	0.015444
5	0.059466	0.068285	0.090399	0.098216	0.118873	0.141577	0.129652	0.117058	0.103710	0.089161
10	0.043113	0.051338	0.069952	0.076767	0.091831	0.107196	0.122246	0.136667	0.149638	0.163637
15	0.029903	0.038193	0.055487	0.063860	0.078803	0.093578	0.107257	0.119126	0.127833	0.135285
20	0.020781	0.029161	0.045670	0.055321	0.070577	0.085612	0.099411	0.111166	0.119413	0.125902
25	0.014802	0.023245	0.039251	0.049761	0.065263	0.080536	0.094521	0.106368	0.114561	0.120792
30	0.010935	0.019419	0.035102	0.046169	0.061835	0.077269	0.091386	0.103308	0.111493	0.117593
35	0.008480	0.016990	0.032468	0.043889	0.059659	0.075195	0.089397	0.101369	0.109550	0.115570
40	0.007280	0.015804	0.031181	0.042775	0.058595	0.074182	0.088425	0.100422	0.108601	0.114582
45	0.005586	0.014128	0.029363	0.041201	0.057094	0.072752	0.087053	0.099083	0.107260	0.113186
50	0.008205	0.013476	0.056046	0.047495	0.060566	0.074376	0.091573	0.107674	0.122930	0.126017

Table 4. Unknown probabilities for bulk service vs. various number of customers in the system when $N = 25$.

n/a	1	2	3	4	5	6	7	8	9	10
0	0.07133259	0.06986017	0.067985	0.06429409	0.059453	0.053631	0.046842	0.038870	0.053924	0.016835
2	0.07466214	0.08671663	0.106182	0.10119590	0.094928	0.087905	0.080516	0.072738	0.096114	0.050279
4	0.06921129	0.07906965	0.095054	0.11587079	0.140425	0.130619	0.120634	0.110713	0.123730	0.085925
6	0.06245147	0.07147820	0.085508	0.10261685	0.122866	0.146223	0.171090	0.157349	0.148828	0.126207
8	0.05525850	0.06402066	0.077402	0.09311807	0.110734	0.129784	0.150316	0.172540	0.178007	0.173461
10	0.04795073	0.05666417	0.069910	0.08524595	0.101995	0.119368	0.137003	0.154796	0.166481	0.187911
12	0.04088728	0.04963191	0.062936	0.07828239	0.094869	0.111744	0.128338	0.144271	0.155465	0.169774
14	0.03436519	0.04316695	0.056595	0.07209020	0.088778	0.105613	0.121915	0.137169	0.145675	0.159300
16	0.03321198	0.04202261	0.055463	0.07096230	0.087624	0.104377	0.120507	0.135453	0.144225	0.156282
18	0.02580714	0.03470241	0.048335	0.06410887	0.081077	0.098105	0.114416	0.129382	0.132943	0.149424
20	0.01992742	0.02889114	0.042679	0.05867876	0.075904	0.093172	0.109661	0.124696	0.123954	0.144318
22	0.01548021	0.02449612	0.038403	0.05457594	0.071999	0.089458	0.106095	0.121203	0.117135	0.140581
24	0.01221983	0.02127417	0.035268	0.05156967	0.069141	0.086742	0.103492	0.118661	0.112124	0.137888

Table 5. System capacity vs. Total average cost.

Threshold Value *a*	System Capacity N	Total average Cost
7	15	8.391925
7	25	8.373304
7	35	8.234628
7	45	8.177738
7	50	8.177716

7.2. Optimal Cost

Simple direct search method (refer Jeyakumar and Arumuganathan [12]) is used to find the optimal policy for a threshold value 'a' to minimize the total average cost. For a fixed system capacity $N = 50$ and the maximum number of customers in a batch of service $b = 10$, total average cost is obtained for 'a' equal to 1 to 10, and is given in the following Table 6.

Also the Figures 2-4 give the relationship between total average cost and threshold value with the following assumption of cost values $C_s = 4$; $C_h = 0.50$; $C_0 = 5$; $C_r = 1$; $C_u = 0.25$; $C_I = 6$ in rupees. From Table 6 and Figure 2, it is observed that the threshold value $a = 7$ gives the minimum total average cost when the maximum capacity of the system is 50. This is used to help the network designers to fix the minimum threshold value $a = 7$ in order to minimise the total average cost.

Figures 3 and 4 give the total average cost for different threshold values ('a' from 1 through 10) and the fixed batch size $b = 10$ with the system capacity $N = 25$ by considering without and with closedown times respectively. It is observed that the total average cost without closedown time is more than the total average cost with closedown time. So, the consideration of closedown times is beneficial for the management to achieve their task with minimal cost.

Table 6. Performance measures and unknown probabilities for various number of customers in the system when $N = 50$.

a	E(Q)	E(B)	E(I)	Total average cost	$\pi^+(n)$		$\omega^+(n)$		$\eta^+(n)$	
					K=1	K=2	K=1	K=2	K=1	K=2
1	6.155834	0.803419	0.062540	11.360755	0.02088149	0.01363176	0.00001283	0.00000753	0.00000345	0.00000191
2	4.683880	0.615260	0.160977	10.399948	0.02262521	0.01480430	0.00013605	0.00007985	0.00004183	0.00002316
3	3.935412	0.540643	0.253536	9.545310	0.02466209	0.01624296	0.00132093	0.00077455	0.00057886	0.00032125
4	3.395916	0.423658	0.296784	8.970882	0.02309866	0.01527146	0.00447060	0.00261994	0.00205506	0.00114275
5	3.837074	0.399854	0.377365	8.535852	0.02060506	0.01362860	0.00871183	0.00510507	0.00368768	0.00205373
6	3.167473	0.331292	0.350043	8.285265	0.01874904	0.01247400	0.02328494	0.01381227	0.01067735	0.00606939
7	**3.121183**	**0.304134**	**0.347942**	**8.177716**	0.01579412	0.01157932	0.03569894	0.02451000	0.01549195	0.01040037
8	4.113374	0.312829	0.432193	8.189498	0.01407076	0.01028356	0.03794790	0.02594085	0.01418776	0.00948849
9	3.634541	0.281583	0.368405	8.213433	0.01294829	0.00949660	0.03250033	0.02225259	0.01280805	0.00859127
10	3.109171	0.248948	0.293930	8.324147	0.01219564	0.00899319	0.02707158	0.01859162	0.01170176	0.00788712

8. Conclusion

This paper gives the cost analysis of finite queues with batch service with multiple vacations and closedown times are analysed using embedded Markov chain and supplementary variable technique. Queue length distribution at various epochs is obtained. Relation between queue length distribution at arbitrary and various epochs are discussed. The measures of interest are also evaluated with numerical illustration. The closedown concept is introduced and the effect with or without closedown model is discussed. Costs analysis is done. Optimal threshold value is obtained for a batch service queue. The model is more effective on account of the inactive timer in the SVC which is activated using closedown time. Cost analysis is used to help the network designers to achieve the minimum cost by properly choosing the threshold value. For further investigation, cost analysis of the other complex models such as *MAP/G(a, b)/1/N* and *BMAP/G(a, b)/1/N* with working vacations and closedown times can be explored.

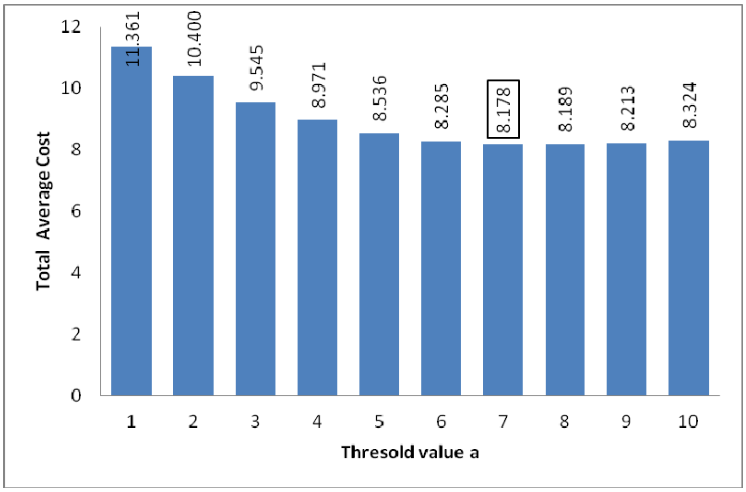

Figure 2. Threshold value vs. Total average cost for $N = 50$ (with closedown times).

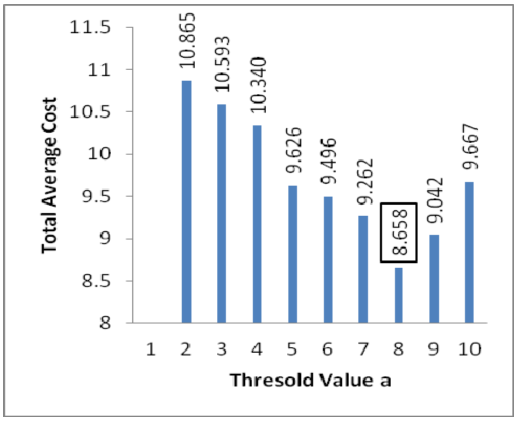

Figure 3. Threshold value vs. Total average cost for $N = 25$ (without closedown times).

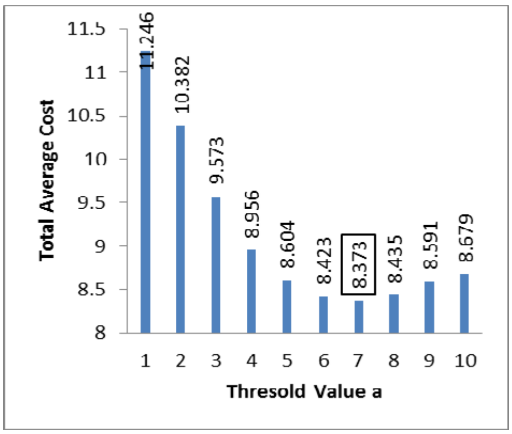

Figure 4. Threshold value vs. Total average cost for $N = 25$ (with closedown times).

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Appendix

Proof of Lemma 1:

Substituting $s=0$ in Equations (14) to (17), we get

$$\boldsymbol{\pi}(0,0)=\boldsymbol{\pi}(0)\mathbf{C}+\sum_{n=a}^{b}\left(\boldsymbol{\pi}(n,0)+\boldsymbol{\eta}(n,0)+\boldsymbol{\omega}(n,0)\right), \tag{a1}$$

$$\boldsymbol{\pi}(n,0)=\boldsymbol{\pi}(n)\mathbf{C}+\boldsymbol{\pi}(n-1)\mathbf{D}+\boldsymbol{\pi}(n+b,0)+\boldsymbol{\eta}(n+b,0)+\boldsymbol{\omega}(n+b,0),\ 1\le n\le N-b, \tag{a2}$$

$$\boldsymbol{\pi}(n,0)=\boldsymbol{\pi}(n)\mathbf{C}+\boldsymbol{\pi}(n-1)\mathbf{D},\ \ N-b+1\le n\le N-1, \tag{a3}$$

$$\boldsymbol{\pi}(N,0)=\boldsymbol{\pi}(N)\left(\mathbf{C}+\mathbf{D}\right)+\boldsymbol{\pi}(N-1)\mathbf{D}. \tag{a4}$$

Post multiplying Equations (a1) to (a4) by $\mathbf{e}$, adding over all possible values of n and using $(\mathbf{C}+\mathbf{D})\mathbf{e}=0$, we get

$$\sum_{n=0}^{a-1}\boldsymbol{\pi}(n,0)\mathbf{e}=\sum_{n=a}^{N}\left(\boldsymbol{\eta}(n,0)+\boldsymbol{\omega}(n,0)\right)\mathbf{e}\,.$$

Proof of Lemma 2:

Substituting $s=0$ in Equations (18) to (21), we get

$$\boldsymbol{\eta}(0,0)=\boldsymbol{\eta}(0)\mathbf{C}+\boldsymbol{\pi}(0,0), \tag{a5}$$

$$\boldsymbol{\eta}(n,0)=\boldsymbol{\eta}(n)\mathbf{C}+\boldsymbol{\eta}(n-1)\mathbf{D}+\boldsymbol{\pi}(n,0),\ 1\le n\le a-1, \tag{a6}$$

$$\boldsymbol{\eta}(n,0)=\boldsymbol{\eta}(n)\mathbf{C}+\boldsymbol{\eta}(n-1)\mathbf{D},\ a\le n\le N-1, \tag{a7}$$

$$\boldsymbol{\eta}(N,0)=\boldsymbol{\eta}(N)\left(\mathbf{C}+\mathbf{D}\right)+\boldsymbol{\eta}(N-1)\mathbf{D}. \tag{a8}$$

Post multiplying Equations (a5) to (a8) by $\mathbf{e}$, adding all possible values of n and using $(\mathbf{C}+\mathbf{D})\mathbf{e}=0$, we get

$$\sum_{n=0}^{N}\boldsymbol{\eta}(n,0)\mathbf{e}=\sum_{n=0}^{a-1}\boldsymbol{\pi}(n,0)\mathbf{e}\,.$$

Proof of Lemma 3:

Substituting $s=0$ in Equations (22) to (25), we get

$$\boldsymbol{\omega}(0,0)=\boldsymbol{\omega}(0)\mathbf{C}+\boldsymbol{\omega}(0,0)+\boldsymbol{\eta}(0,0), \tag{a9}$$

$$\boldsymbol{\omega}(n,0)=\boldsymbol{\omega}(n)\mathbf{C}+\boldsymbol{\omega}(n-1)\mathbf{D}+\boldsymbol{\omega}(n,0)+\boldsymbol{\eta}(n,0),\ 1\le n\le a-1, \tag{a10}$$

$$\boldsymbol{\omega}(n,0)=\boldsymbol{\omega}(n)\mathbf{C}+\boldsymbol{\omega}(n-1)\mathbf{D},\ a\le n\le N-1, \tag{a11}$$

$$\boldsymbol{\omega}(N,0)=\boldsymbol{\omega}(N)\left(\mathbf{C}+\mathbf{D}\right)+\boldsymbol{\omega}(N-1)\mathbf{D}. \tag{a12}$$

Post multiplying Equations (a9) to (a12) by $\mathbf{e}$, adding all possible values of n and using $(\mathbf{C}+\mathbf{D})\mathbf{e}=0$, we get

$$\sum_{n=a}^{N}\boldsymbol{\omega}(n,0)\mathbf{e}=\sum_{n=0}^{a-1}\left(\boldsymbol{\eta}(n,0)\right)\mathbf{e}\,.$$

Proof of Lemma 4:

Differentiating Equations (14) to (25) with respect to s and substituting $s=0$ in the these equations, we get

$$\boldsymbol{\pi}(0)=\boldsymbol{\pi}'(0)\mathbf{C}+\theta_s\sum_{n=a}^{b}\left(\boldsymbol{\pi}(n,0)+\boldsymbol{\eta}(n,0)+\boldsymbol{\omega}(n,0)\right), \tag{a13}$$

$$\boldsymbol{\pi}(n)=\boldsymbol{\pi}'(n)\mathbf{C}+\boldsymbol{\pi}'(n-1)\mathbf{D}+\theta_s\left(\boldsymbol{\pi}(n+b,0)+\boldsymbol{\eta}(n+b,0)+\boldsymbol{\omega}(n+b,0)\right),\ 1\le n\le N-b, \tag{a14}$$

$$\boldsymbol{\pi}(n)=\boldsymbol{\pi}'(n)\mathbf{C}+\boldsymbol{\pi}'(n-1)\mathbf{D},\ N-\mathrm{b}+1\le n\le N-1, \tag{a15}$$

$$\boldsymbol{\pi}(N)=\boldsymbol{\pi}'(N)(\mathbf{C}+\mathbf{D})+\boldsymbol{\pi}'(N-1)\mathbf{D}, \tag{a16}$$

$$\boldsymbol{\eta}(0)=\boldsymbol{\eta}'(0)\mathbf{C}+\theta_u(\boldsymbol{\pi}(0,0)), \tag{a17}$$

$$\boldsymbol{\eta}(n)=\boldsymbol{\eta}'(n)\mathbf{C}+\boldsymbol{\eta}'(n-1)\mathbf{D}+\theta_u(\boldsymbol{\pi}(n,0)),\ 1\le n\le a-1, \tag{a18}$$

$$\boldsymbol{\eta}(n)=\boldsymbol{\eta}'(n)\mathbf{C}+\boldsymbol{\eta}'(n-1)\mathbf{D},\ a\le n\le N-1, \tag{a19}$$

$$\boldsymbol{\eta}(N)=\boldsymbol{\eta}'(N)(\mathbf{C}+\mathbf{D})+\boldsymbol{\eta}'(N-1)\mathbf{D}, \tag{a20}$$

$$\boldsymbol{\omega}(0)=\boldsymbol{\omega}'(0)\mathbf{C}+\theta_v\left(\boldsymbol{\eta}(0,0)+\boldsymbol{\omega}(0,0)\right), \tag{a21}$$

$$\boldsymbol{\omega}(n)=\boldsymbol{\omega}'(n)\mathbf{C}+\boldsymbol{\omega}'(n-1)\mathbf{D}+\theta_v\left(\boldsymbol{\eta}(n,0)+\boldsymbol{\omega}(n,0)\right),\ 1\le n\le a-1, \tag{a22}$$

$$\boldsymbol{\omega}(n)=\boldsymbol{\omega}'(n)\mathbf{C}+\boldsymbol{\omega}'(n-1)\mathbf{D},\ a\le n\le N-1, \tag{a23}$$

$$\boldsymbol{\omega}(N)=\boldsymbol{\omega}'(N)\left(\mathbf{C}+\mathbf{D}\right)+\boldsymbol{\omega}'(N-1)\mathbf{D}. \tag{a24}$$

Post multiplying Equations (a13) to (a16) by $\mathbf{e}$, adding all possible values of n and using $(\mathbf{C}+\mathbf{D})\mathbf{e}=0$ and also the Lemma 1, we get

$$\sum_{n=0}^{N}\boldsymbol{\pi}(n)\mathbf{e}=\theta_s\left[\sum_{n=0}^{N}\boldsymbol{\pi}(n,0)\mathbf{e}\right]=\rho_1'.$$

Post multiplying Equations (a17) to (a20) by $\mathbf{e}$, adding all possible values of n and using $(\mathbf{C}+\mathbf{D})\mathbf{e}=0$ and also the Lemma 2, we get

$$\sum_{n=0}^{N}\boldsymbol{\eta}(n)\mathbf{e}=\theta_u\left[\sum_{n=0}^{N}\boldsymbol{\eta}(n,0)\mathbf{e}\right]=1-\rho_1'-\rho_2'.$$

Post multiplying Equations (a21) to (a24) by $\mathbf{e}$, adding all possible values of n and using $(\mathbf{C}+\mathbf{D})\mathbf{e}=0$ and also the Lemma 3, we get

$$\sum_{n=0}^{N}\boldsymbol{\omega}(n)\mathbf{e}=\theta_v\left[\sum_{n=0}^{N}\boldsymbol{\omega}(n,0)\mathbf{e}\right]=\rho_2'.$$

Proof of Lemma 5:

As $\theta_s\left[\sum_{n=0}^{N}\boldsymbol{\pi}(n,0)\mathbf{e}\right]=\rho_1'$, $\theta_v\left[\sum_{n=0}^{N}\boldsymbol{\omega}(n,0)\mathbf{e}\right]=\rho_2'$, $\theta_u\left[\sum_{n=0}^{N}\boldsymbol{\eta}(n,0)\mathbf{e}\right]=1-\rho_1'-\rho_2'$.

Using these equations in $\sigma=\sum_{n=0}^{N}\left(\boldsymbol{\pi}(n,0)+\boldsymbol{\omega}(n,0)+\boldsymbol{\eta}(n,0)\right)\mathbf{e}$,

$$\sigma=\frac{\rho_1'}{\theta_s}+\frac{\rho_2'}{\theta_v}+\frac{1-\rho_1'-\rho_2'}{\theta_u},\qquad \sigma=\frac{\theta_v\theta_u\rho_1'+\theta_s\theta_u\rho_2'+\theta_v\theta_s(1-\rho_1'-\rho_2')}{\theta_v\theta_u\theta_s}.$$

Authors' Biographies:

A. Senthil Vadivu is currently Assistant Professor in the Department of Mathematics, PSG Institute of Technology and Applied Research, Coimbatore, India. She received his M.Sc. and M. Phil. degrees in Mathematics from Bharathiar University, Coimbatore, India in 2002 and 2005 respectively and Ph.D. degree in Mathematics from Anna University, Coimbatore, India. Her field of interest includes Queuing theory.

R. Arumuganathan is currently Professor and Head in the Department of Mathematics, PSG College of Technology, Coimbatore, India. He received M. Sc. and M. Phil degrees in Mathematics from Madurai Kamarajar University, Madurai, India in 1985 and 1986 respectively and Ph.D degree in Mathematics from Bharathiar University, Coimbatore, India. His fields of interest include queueing theory, reliability theory and wavelet transforms. He has publications in international journals like Computers and Operations Research, Applied Mathematical Modelling, Asia Pacific Journal of Operational Research, OPSEARCH and Int. J. Information and Management Sciences, etc. He has given invited talks in several international conferences.