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Decomposition of hypercube graphs into paths and cycles having k edges

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ABSTRACT

For an even graph G and positive integer p, q, and k, the pair (p, q) is an admissible pair if $(p+q)k = |E(G)|$. If a graph G admits a decomposition into p copies of P_{k+1}, the path of length k, and q copies of C_k, the cycle of length k, for every admissible pair (p, q), then G has a $\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposition. In this paper, we give necessary and sufficient conditions for the existence of a $\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposition of n-dimensional hypercube graphs Q_n when n is even, $k \geq 4$, and $n \equiv 0 \pmod{k}$.

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1. Introduction

The graphs presented in this paper are simple, finite and undirected. Let P_{k+1} be a *path* of length k and C_k be a *cycle* of length k. Here $u_1u_2 \dots u_{k-1}u_k$ represents the path P_k and $(u_1u_2 \dots u_ku_1)$ indicates the cycle C_k respectively with vertices $u_1, u_2, \dots, u_k$ and edges $u_1u_2, u_2u_3, \dots, u_{k-1}u_k, u_ku_1$. For any integer $\psi > 0$, ψG represents a graph consisting of ψ edge-disjoint copies of G. Better clarity on the terms of standard graph theory shall be referred in ref.[1] The n-dimensional hypercube Q_n is the graph with $V(Q_n) = \{0,1\}^n$ and edges between pairs of vertices that differ in exactly one co-ordinate. The tuple is named as *odd* or *even* when the number of $1'$s is odd or even.

A decomposition of a graph H is a collection of *edge-disjoint* subgraphs $G_1, G_2, \dots, G_r$ of H such that every edge of H is in exactly one G_i. When decomposing the H, if all subgraphs are isomorphic to graph G, then H can be decomposed into G which is defined as *G-decomposition*. In this context, we say that H has a *$\{pG_1, qG_2\}$-decomposition*, when there is a decomposition of p and q copies of G_1 and G_2, respectively. For integers p and q, the pair (p, q) is an admissible pair for the even graph H, that is, every vertex has an even degree, if $(p+q)k = |E(H)|$. If H admits a decomposition into p copies of P_{k+1} and q copies of C_k for every admissible pair (p, q), then H has a *$\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposition*.

There are number of extensive analysis of the n-dimensional hypercube graph Q_n exist. A detailed analysis of hypercube is developed by Harary et al.[2] and also examined about the properties of coloring, distance, genus and domination of a graph. In ref.,[3] Horak et al. proved that the graph Q_n can be decomposed into any graph G of size n each of whose block either an even cycle or an edge, and it will be further decomposed into any set of six trees of size n. Anick and Ramras[4] proved that Q_n admits a P_m-decomposition for odd n if and only if $m \leq n$ and $m \mid n2^{n-1}$. Tapadia et al.[5] proved that for even n and m such that $2^m < n$, Q_n is decomposable into paths of length at most $2^m n$. Axenovich et al.[6] proved that when n is even, Q_n can be decomposed into long cycles from which it follows that there are decompositions of such hypercube into long paths.

Though, study on $\{G_1, G_2\}_{\{p,q\}}$-decomposition of complete graph and complete bipartite graphs already exist, Shyu has derived that the K_n admitted $\{P_5, C_4\}_{\{p,q\}}$-decomposition with necessary and sufficient conditions in Ref.[7] The tensor and cartesian product of paths, cycles and K_r graphs are decomposed to $\{C_4, E_2\}$ with necessary and suitable conditions were obtained by Abueida and Devan,[8] where E_2 denotes the 4-vertex graph having two disjoint edges. The existence of $\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposition of K_n and $K_{m,n}$ with necessary and sufficient conditions are obtained by Jeevadoss and Muthusamy.[9] They extended their study of decomposition to complete bipartite multi-graphs $K_{m,n}(\lambda)$ in Ref.[10] The presence of $\{P_5, C_4\}_{\{p,q\}}$-decomposition of cartesian as well as tensor product of complete graphs with necessary and sufficient conditions are obtained by Jeevadoss and Muthusamy.[11] Recently many authors are working in the field of paths and cycles decomposition,[12,13] paths and stars decomposition,[14–16] cycles and stars decomposition[17,18] and paths, cycles and stars decomposition[19] problems in graphs. Saranya and Jeevadoss[20] proved that the graph Q_n is $\{P_5, C_4\}_{\{p,q\}}$-decomposable.

In this paper, the existence of $\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposition of Q_n has been proved for $n \equiv 0 \pmod{k}$ with $n, k \geq 4$, where n and k are even.

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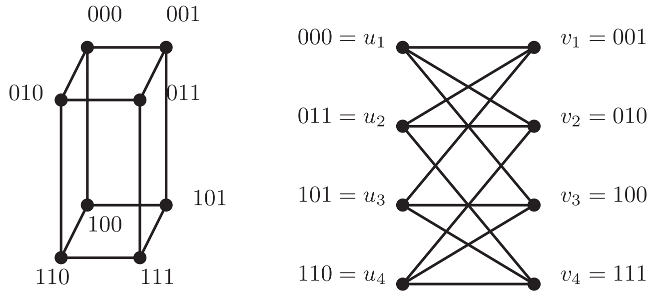

Figure 1. 3-dimensional hypercube graph.

2. Preliminaries

The n-dimensional hypercube graph represented as the n-regular bipartite graph. For example, the graph Q_3 is represented as a 3-regular bipartite graph in Figure 1, which has two partition namely U and V whose vertex labels are (u_1, u_2, u_3, u_4) and (v_1, v_2, v_3, v_4), respectively.

Generate all possible binary n-tuples for the n-dimensional hypercube graph. Arrange the binary tuples in ascending order by converting them to decimal numbers or by sorting them lexicographically. Classify all the binary n-tuples as *odd* or *even* based on the number of $1'$s in the tuple. If the number of $1'$s is odd, the tuple is *odd*; if it is even, the tuple is *even*. Consider the even tuples as U partite vertices and the odd tuples as V partite vertices of the n-regular bipartite graph. Assign the labels $u_1, u_2, \ldots, u_{2^{n-1}}$ to the U partite vertices in order and $v_1, v_2, \ldots, v_{2^{n-1}}$ to the V partite vertices in order. The U and V partite vertices are adjacent if it satisfies the following conditions.

u_i is adjacent to

$$\begin{cases} v_i & \textit{and} \\ \begin{cases} v_{i+1}, & \textit{if } i \textit{ is odd} \\ v_{i-1}, & \textit{if } i \textit{ is even} \end{cases} & \textit{and} \\ \begin{cases} v_{i+2^j}, & \textit{if } i \equiv 1, 2, \ldots, 2^j \pmod{2^{j+1}} \\ v_{i-2^j}, & \textit{if } i \equiv 2^j+1, 2^j+2, \ldots, 2^{j+1} \pmod{2^{j+1}} \end{cases} & \end{cases}$$

for j, if $1 \le j \le (n-2)$ and for each i, $1 \le i \le 2^{n-1}$.

When $n = 4$, $j = 1, 2$ and $1 \le i \le 8$, the adjacent vertices with respect to i are (Figure 2)

u_1 is adjacent to v_1, v_2, v_3, v_5
u_2 is adjacent to v_2, v_1, v_4, v_6
u_3 is adjacent to v_3, v_4, v_1, v_7
u_4 is adjacent to v_4, v_3, v_2, v_8
u_5 is adjacent to v_5, v_6, v_7, v_1
u_6 is adjacent to v_6, v_5, v_8, v_2
u_7 is adjacent to v_7, v_8, v_5, v_3
u_8 is adjacent to v_8, v_7, v_6, v_4

When $n = 5$, $1 \le j \le 3$ and $1 \le i \le 16$, the adjacent vertices with respect to i are

u_1 is adjacent to v_1, v_2, v_3, v_5, v_9
u_2 is adjacent to $v_2, v_1, v_4, v_6, v_{10}$
u_3 is adjacent to $v_3, v_4, v_1, v_7, v_{11}$

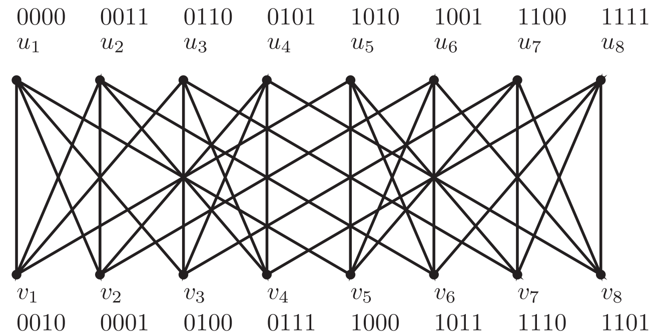

Figure 2. 4-dimensional hypercube graph Q_4 represented as 4−regular bipartite graph $H_{8,8}$.

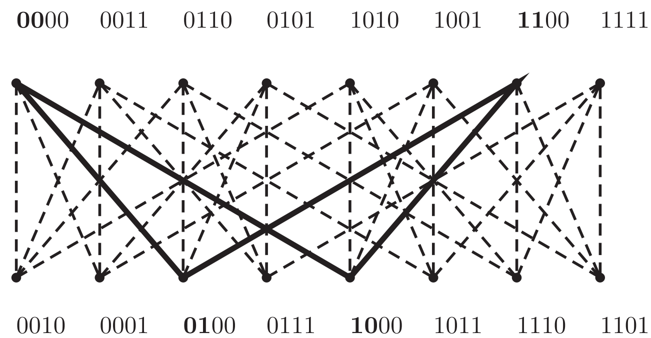

Figure 3. Q_2-decomposition of Q_4.

u_4 is adjacent to $v_4, v_3, v_2, v_8, v_{12}$
u_5 is adjacent to $v_5, v_6, v_7, v_1, v_{13}$
u_6 is adjacent to $v_6, v_5, v_8, v_2, v_{14}$
u_7 is adjacent to $v_7, v_8, v_5, v_3, v_{15}$
u_8 is adjacent to $v_8, v_7, v_6, v_4, v_{16}$
u_9 is adjacent to $v_9, v_{10}, v_{11}, v_{13}, v_1$
u_{10} is adjacent to $v_{10}, v_9, v_{12}, v_{14}, v_2$
u_{11} is adjacent to $v_{11}, v_{12}, v_9, v_{15}, v_3$
u_{12} is adjacent to $v_{12}, v_{11}, v_{10}, v_{16}, v_4$
u_{13} is adjacent to $v_{13}, v_{14}, v_{15}, v_9, v_5$
u_{14} is adjacent to $v_{14}, v_{13}, v_{16}, v_{10}, v_6$
u_{15} is adjacent to $v_{15}, v_{16}, v_{13}, v_{11}, v_7$
u_{16} is adjacent to $v_{16}, v_{15}, v_{14}, v_{12}, v_8$

Freezing refers to fixing the selected co-ordinates in n-tuples and taking all possible combinations of the remaining co-ordinates. For example, consider all 4-tuples and partition it into two parts. First 2-tuples as part I say b_1 and last 2-tuples as part II say b_2. There are four possible cases of 2-tuples 00, 01, 10, and 11. Freeze b_2 as one of its possible cases say 00 and take all possible case of b_1, we get $S = \{00\mathbf{00}, 01\mathbf{00}, 10\mathbf{00}, 11\mathbf{00}\}$. The induced subgraph of the S is isomorphic to the graph Q_2 as shown in Figure 3. Similarly, perform the same process for all the possible cases of b_2, 4 copies of Q_2 can be obtained. Then, freeze b_1 and take all the possible cases of b_2, 4 copies of Q_2 can be obtained.

The following construction is useful for the main theorem.

Construction 1. [9] Consider two cycles $C_k^{(1)}$ and $C_k^{(2)}$ of length k, where $C_k^{(1)} = (u_1 u_2 u_3 \cdots u_k u_1)$ and $C_k^{(2)} = (v_1 v_2 v_3 \cdots v_k v_1)$. If x is a common vertex of $C_k^{(1)}$ and $C_k^{(2)}$ such that at least one neighbor of vertex x from each cycle (say, u_i and

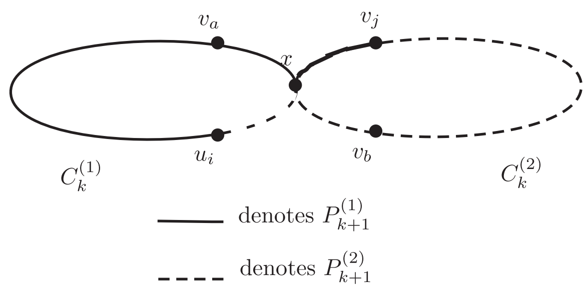

Figure 4. ${C_k}^{(1)} \cup {C_k}^{(2)} = P_{k+1}^{(1)} \cup P_{k+1}^{(2)}$.

v_j) does not belong to other cycle. Then there will be two edge-disjoint paths of length k, say ${P_{k+1}}^{(1)}$ and ${P_{k+1}}^{(2)}$ from ${C_k}^{(1)}$ and ${C_k}^{(2)}$ as shown in Figure 4, where ${P_{k+1}}^{(1)} = ({C_k}^{(1)} - xu_i) \cup xv_j$, ${P_{k+1}}^{(2)} = ({C_k}^{(2)} - xv_j) \cup xu_i$.

Remark 2.1. If G and H have a $\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposition then $G \cup H$ has such a decomposition.

3. Main theorem

In this section, we prove that the graph Q_n has a $\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposition.

Theorem 3.1. Let p, q be nonnegative integers and n, k be positive even integers with $k \geq 4$. If the graph Q_n has a $\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposition, then $n2^{n-1} = k(p+q)$ and $p \neq 1$.

Proof. Obviously, if Q_n can be decomposed into p copies of P_{k+1} and q copies of C_k, then $k(p+q) = |E(Q_n)| = n2^{n-1}$. On the contrary, suppose that $p = 1$. Let P denotes a path of length k in the decomposition. It implies that the end vertices of P have odd degree in $Q_n - E(P)$. Therefore, $Q_n - E(P)$ cannot be decomposed into cycles, a contradiction. □

Theorem 3.2. The graph Q_n has a Q_k-decomposition for all $n, k \geq 4$ and $n \equiv 0 \pmod{k}$.

Proof. The vertex set $V(Q_n)$ is the set of all binary n-tuples. Partition the n-tuples into $\frac{n}{k}$ parts and label them by b_1,b_2, …,$b_{\frac{n}{k}}$. The induced subgraph is obtained by freezing all the parts except b_1, that is b_2, b_3, …,$b_{\frac{n}{k}}$ as one of its possible case, there are 2^{n-k} possible cases of $(n-k)$-tuples, which is isomorphic to Q_k. Similarly, the same process is performed for exception of b_2 then, b_3 upto $b_{\frac{n}{k}}$ sequentially. As a result $\frac{n}{k}2^{n-k}$ edge-disjoint copies of Q_k are obtained. □

Theorem 3.3. Let p, q be non-negative integers, n be a positive even integer with $n \geq 4$ and $p \neq 1$. Then the graph Q_n has a $\{P_{n+1}, C_n\}_{\{p,q\}}$-decomposition.

Proof. Represent the n-dimensional hypercube graph Q_n as a n-regular bipartite graph, which has two partition namely U and V whose vertex labels are $(u_1, u_2, \ldots, u_{2^{n-1}})$ and $(v_1, v_2, \ldots, v_{2^{n-1}})$ respectively. Divide the each U and V partite into 4 equal sub-partite whose labels are (U_1, U_2, U_3, U_4) and (V_1, V_2, V_3, V_4), respectively. Type I cycles are cycles which covers the vertices from more than one sub-partite of V as shown in Figure 5. Type II cycles are cycles which covers the vertices from exactly one sub-partite of V as shown in Figure 6. Thus the result in two cases prove $n \equiv 0 \pmod{4}$ and $n \equiv 2 \pmod{4}$.

Case (i) : $n \equiv 0 \pmod{4}$
Type I cycles:
For each hypercube graph Q_n, a sign table can be generated of order $(2^{\frac{n}{2}-1} \times n)$, whose each cell is identified by $r_s c_t$ (sth row and tth column). The sign table is divided vertically into two halves, say Part-I and Part-II. Part-I is from c_1 to $c_{\frac{n}{2}}$. Part-II is from $c_{\frac{n}{2}+1}$ to c_n. In Part-I, positive sign is assigned in $c_{\frac{n}{2}}$ and for the remaining cells, the $(\frac{n}{2}-1)$-tuples binary string is generated, then 0 and 1 is replaced by $+$ and $-$, respectively. Consider the negation of Part-I as Part-II. (See Example 3.1)

The following table gives the base numbers and names which are used to generate 2^{n-2} copies of C_n. (See Example 3.2)

Base numbers	2^{n-2}	2^{n-3}	⋯	$2^{\frac{n}{2}-1}$	2^{n-2}	2^{n-3}	⋯	$2^{\frac{n}{2}-1}$
Names	δ_1	δ_2	⋯	$\delta_{\frac{n}{2}}$	$\delta_{\frac{n}{2}+1}$	$\delta_{\frac{n}{2}+2}$	⋯	δ_n

Following construction is used to generate C_n.

$$(u_i \quad v_{i+\delta_1.r_sc_1} \quad u_{i+\delta_1.r_sc_1+\delta_2.r_sc_2} \quad v_{i+\delta_1.r_sc_1+\delta_2.r_sc_2+\delta_3.r_sc_3} \cdots$$
$$u_{i+\delta_1.r_sc_1+\delta_2.r_sc_2+\delta_3.r_sc_3 \ldots,+\delta_n.r_sc_n})$$

where $i \equiv j \pmod{2^{\frac{n}{2}}}$ for $1 \leq j \leq 2^{\frac{n}{2}-1}$ and $1 \leq i \leq 2^{n-1}$, for $s = q+1$ and $i = q \times 2^{\frac{n}{2}} + j$. For each j $(1 \leq j \leq 2^{\frac{n}{2}-1})$ there are $\frac{2^{n-1}}{2^{\frac{n}{2}}}$ different i values while the condition $i \equiv j \pmod{2^{\frac{n}{2}}}$ satisfies. Hence this construction gives $2^{\frac{n}{2}-1} \times \frac{2^{n-1}}{2^{\frac{n}{2}}} = 2^{n-2}$ copies of C_n.

Type II cycles:

For each hypercube graph Q_n, a sign table can be generated in the order $(2^{\frac{n}{2}-1} \times n)$, whose each cell is identified by $r_s c_t$ (sth row and tth column). The sign table is divided vertically into two halves, say Part-I and Part-II. Part-I is from c_1 to $c_{\frac{n}{2}}$, Part-II is from $c_{\frac{n}{2}+1}$ to c_n. In Part-I, positive sign is assigned in c_1 and $c_{\frac{n}{2}}$ and for the remaining cells, the $(\frac{n}{2}-2)$-tuples binary string is generated, then 0 and 1 is replaced by $+$ and $-$, respectively. Consider the negation of Part-I as Part-II. (See Example 3.3)

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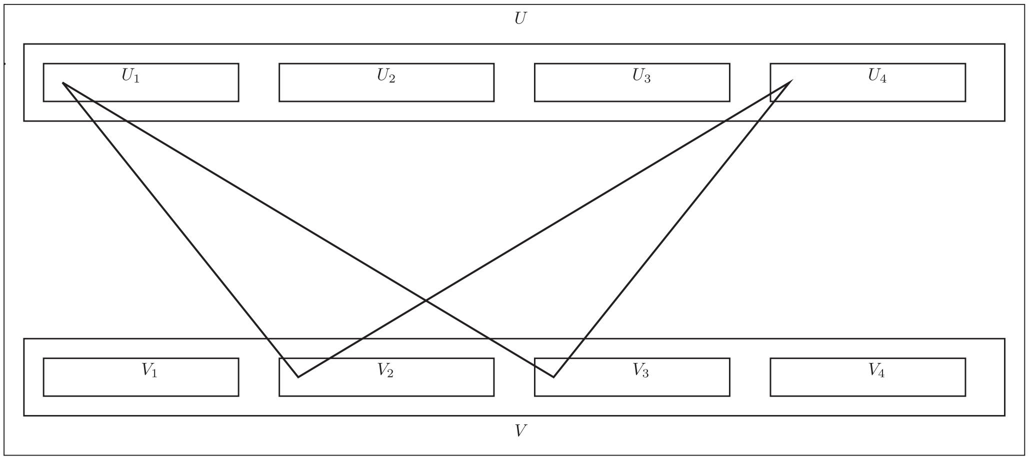

Figure 5. Type I cycle.

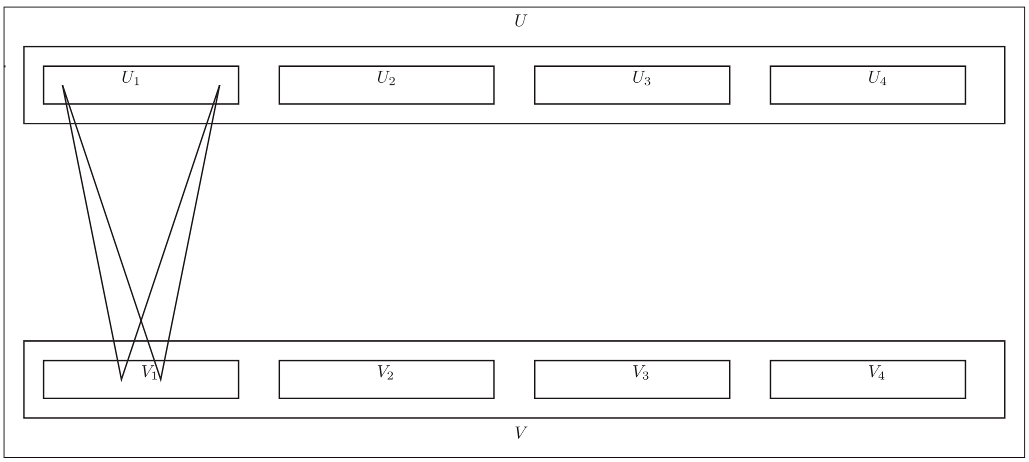

Figure 6. Type II cycle.

Base numbers	$2^{\frac{n}{2}-2}$	$2^{\frac{n}{2}-3}$	$\cdots$	2^0	0	$2^{\frac{n}{2}-2}$	$2^{\frac{n}{2}-3}$	$\cdots$	2^0	0
Names	λ_1	λ_2	$\cdots$	$\lambda_{\frac{n}{2}-1}$	$\lambda_{\frac{n}{2}}$	$\lambda_{\frac{n}{2}+1}$	$\lambda_{\frac{n}{2}+2}$	$\cdots$	λ_{n-1}	λ_n

The following construction is used to generate C_n.

$$(u_i \quad v_{i+\lambda_1.r_sc_1} \quad u_{i+\lambda_1.r_sc_1+\lambda_2.r_sc_2} \quad .v_{i+\lambda_1.r_sc_1+\lambda_2.r_sc_2+\lambda_3.r_sc_3} \cdots u_{i+\lambda_1.r_sc_1+\lambda_2.r_sc_2+\lambda_3.r_sc_3\ ,\cdots,+\lambda_n.r_sc_n})$$

where $i \equiv j \pmod{2^{\frac{n}{2}-1}}$, $1 \le j \le 2^{\frac{n}{2}-2}$ and $1 \le i \le 2^{n-1}$, for $s = j$. For each $j\,(1 \le j \le 2^{\frac{n}{2}-2})$ there are $\frac{2^{n-1}}{2^{\frac{n}{2}-1}}$ different i values while the condition $i \equiv j \pmod{2^{\frac{n}{2}-1}}$ satisfies. So this construction gives $2^{\frac{n}{2}-2} \times \frac{2^{n-1}}{2^{\frac{n}{2}-1}} = 2^{n-2}$ copies of C_n. Hence, from Case (i), 2^{n-2} copies of Type I cycles + 2^{n-2} copies of Type II cycles = 2^{n-1} copies of C_n is obtained.

Case (ii) : $n \equiv 2 \pmod 4$

The graph Q_n is decomposed into $4Q_{n-2}$ as represented in Figure 7. Each part stands for copies of Q_{n-2} induced by n-tuples by freezing the last two coordinates. The graph Q_{n-2} can be decomposed into C_{n-2} by Case (i). For the vertex g from the Q_{n-2}, it is written g_{ij} whose last two coordinates are ij. Consider a cycle F which is of the form of Type I cycles. One of those is depicted in Figure 7. Here fag denotes the upper part of the cycle and fbg denotes the lower part of the cycle. I_1, I_2, I_3 and I_4 are four cycles of length n, such as

$$I_1 = f_{00}a_{00}g_{00}g_{01}a_{01}f_{01}f_{00}$$
$$I_2 = f_{00}b_{00}g_{00}g_{10}b_{10}f_{10}f_{00}$$
$$I_3 = f_{10}a_{10}g_{10}g_{11}a_{11}f_{11}f_{10}$$
$$I_4 = f_{01}b_{01}g_{01}g_{11}b_{11}f_{11}f_{01}$$

where $f = u_i$ and $g = u_{i+\delta_1.r_sc_1+\delta_2.r_sc_2+\cdots+\delta_{\frac{n}{2}}.r_sc_{\frac{n}{2}}}$ in Type I cycles and $f = v_{i+\lambda_1.r_sc_1}$ and $g = v_{i+\lambda_1.r_sc_1+\lambda_2.r_sc_2+\cdots+\lambda_{\frac{n}{2}}.r_sc_{\frac{n}{2}}+\lambda_{\frac{n}{2}+1}.r_sc_{\frac{n}{2}+1}}$ in Type II cycles. Hence from Case (i) and Case (ii), Q_n is decomposed into 2^{n-1} copies of C_n.

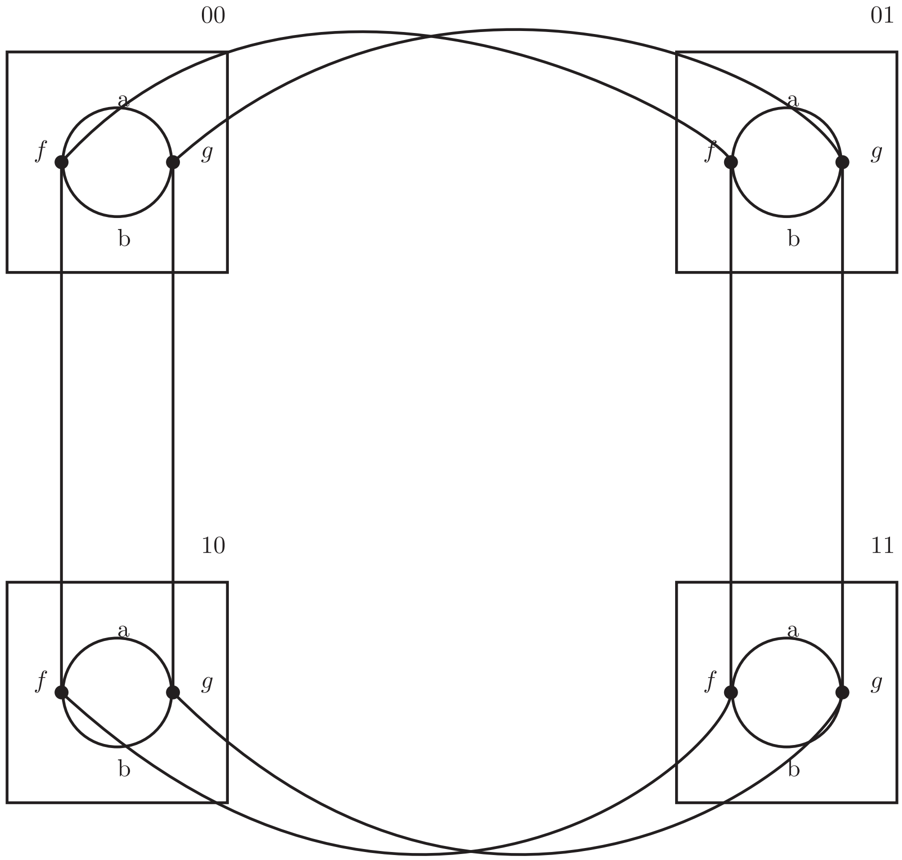

Figure 7. Q_n represented as 4 copies of Q_{n-2}.

In Case (i), consider the vertices u_i and $u_{i+\delta_1.r_sc_1+\delta_2.r_sc_2+\cdots+\delta_{\frac{n}{2}}.r_sc_{\frac{n}{2}}}$ in Type I cycles, u_i and $u_{i+\lambda_1.r_sc_1+\lambda_2.r_sc_2+\cdots+\lambda_{\frac{n}{2}}.r_sc_{\frac{n}{2}}}$ in Type II cycles cover all the U partite vertices exactly once and Type I and Type II cycles are edge-disjoint. Therefore, these cycles have at least one common vertex and at least one neighbor vertex of that common vertex which does not belong to other cycles. By using Construction 3, the same number of edge-disjoint paths of same length can be obtained.

In Case (ii), the cycles I_1, I_2, I_3, and I_4 are edge-disjoint cycles, here f_{00} is a common vertex of I_1 and I_2 and f_{10} is a common vertex of I_3 and I_4, then a and b are neighbor vertex of the common vertex from each cycles which does not belong to the other cycles. By Construction 1, these cycles decomposed into same number of paths. Hence the n-dimensional hypercube graph Q_n is $\{P_{n+1}, C_n\}_{\{p,q\}}$-decomposable. □

Example 3.1. When $n = 8$ the sign table for type I cycles is

+	+	+	+	-	-	-	-
+	+	-	+	-	-	+	-
+	-	+	+	-	+	-	-
+	-	-	+	-	+	+	-
-	+	+	+	+	-	-	-
-	+	-	+	+	-	+	-
-	-	+	+	+	+	-	-
-	-	-	+	+	+	+	-

Example 3.2. When $n = 8$ the base number and its names for Type I cycles are

Base number	64	32	16	8	64	32	16	8
Name	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6	δ_7	δ_8

Example 3.3. When $n = 8$ the sign table for Type II cycles is

+	+	+	+	-	-	-	-
+	+	-	+	-	-	+	-
+	-	+	+	-	+	-	-
+	-	-	+	-	+	+	-
+	+	+	+	-	-	-	-
+	+	-	+	-	-	+	-
+	-	+	+	-	+	-	-
+	-	-	+	-	+	+	-

Example 3.4. When $n = 8$ the base number and its names for Type II cycles are

Base number	4	2	1	0	4	2	1	0
Name	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8

Theorem 3.4. Let p, q be nonnegative integers and $n, k \geq 4$ be positive even integers with $p \neq 1$. If $n \equiv 0 \pmod{k}$, then the graph Q_n has a $\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposition.

Proof. By Theorems 3.2 and 3.3, the graph Q_n is $\{P_{k+1}, C_k\}_{\{p,q\}}$-decomposable for $n \equiv 0 \pmod{k}$ with $n \geq 4$ and $4 \leq k \leq n$, where n and k are even. □

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Appendices

The Python program is given in the link : *https://github.com/Saranyadgraph/saranya.git*

The output of the program provides the required bit strings and their corresponding labels of Q_n for $n \equiv 0 \pmod 4$. However, it gives the adjacent vertices of each vertex in the graph. After that it gives the 2^{n-1} copies of C_n. Finally it gives the common vertex and their corresponding cycles