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A NOTE ON $P(r,m)\Gamma$-SEMINEAR-RINGS

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Abstract: In this paper, we consider a $P(r,m)$ Γ - seminear-rings and obtain equivalent conditions for such Γ - seminear-rings. we also obtain several characterizations of a $P(r,m)$ Γ - seminear-rings which are admitting mate functions.

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1. Introduction

The cocept of seminear-rings was introduced by B. V. Rootselaar in 1962 [15]. It is known that seminear-rings are common generalization of nearrings and semirings. The purpose of this paper is to establish the concept $P(r,m)$ Γ - seminear-rings and obtain some of their properties. In Section 2, we give preliminaries of Γ - seminear-rings which are used in the subsequent Sections. In Section 3, we give examples of $P(r,m)$ Γ - seminear-rings.

In Section 4, we discuss the properties of $P(1,1)$,$P(1,2)$ and $P(2,1)$ Γ - seminear-ring and obtain the necessary and sufficient conditions under which such Γ - seminear-rings posses mate function.

In Section 5, we prove equivalent conditions for $P(r,m)$ Γ - seminear-rings. we also obtain several characterisations of a $P(r,m)$ Γ - seminear-ring admitting mate functions, it is interesting to observe that a Γ - seminear-ring with a mate

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function is a $P(r,m)$ Γ - seminear-ring for all positive integers r and m if and only if it is a $P(1,2)$ Γ - seminear-ring. Throughout this paper, by a Γ - seminear-ring we mean a right Γ - seminear-ring with an absorbing zero. We write ab to denote the product $a.b$ for any two elements a, b in R.

2. Preliminaries

Definition 2.1. An algebraic structure $(R,+,.)$ is said to be a seminear-ring if

(i) $(R,+)$ is a semi group

(ii) $(R,.)$ is a semi group

(iii) $(a+b)c = ac+bc$ for all $a,b,c \in R$.

Definition 2.2. Let R be an additive semigroup and Γ a nonempty set. Then R is called a right Γ - seminear-ring if there exists a mapping $R \times \Gamma \times R \to R$ satisfying the following conditions:

(i) $(a+b)\gamma c = a\gamma c + b\gamma c$

(ii) $(a\gamma b)\beta c = a\gamma(b\beta c)$ for all $a,b,c \in R$ and $\gamma,\beta \in \Gamma$.

Definition 2.3. Let R be a Γ - seminear-ring under the mapping f : $R \times \Gamma \times R \to R$. a subsemigroup A of R is called a sub Γ - seminear-ring of R if A is a Γ - seminear-ring under the restriction of f to $A \times \Gamma \times A$.

Definition 2.4. A right Γ - seminear-ring R is said to have an absorbing zero $'0'$ if

(i) $a+0 = 0+a = a$

(ii) $a\gamma 0 = 0\gamma a = 0$, hold for all $a \in R$ and $\gamma \in \Gamma$

Definition 2.5. $(R,+,\Gamma)$ is a right Γ - seminear-field if

(i) $(R,+)$ is a semigroup

(ii) (R^*,Γ) is a group (R^* is R without addition zero, if it has one)

(iii) $(a+b)\gamma c = a\gamma c + b\gamma c$ for all $a,b,c \in R$,$\gamma \in \Gamma$

Definition 2.6. A Γ - seminear-ring homomorphism between two right Γ - seminear -ring R and R' is a map $\phi : R \to R'$ satisfying

(i) $\phi(a+b) = \phi(a) + \phi(b)$

(ii) $\phi(a\gamma b) = \phi(a)\gamma\phi(b)$ for all $a, b \in R$,and $\gamma \in \Gamma$

Definition 2.7. An ideal of a seminear-ring R is defined to be the kernel of a homomorphism of R. A left ideal of a seminear-ring R is defined to be an R-kernel of the R-semigroup $(R,+)$.

Definition 2.8. An element $a \in R$ is said to be

(i) idempotent if $a\gamma a = a$.

(ii) nilpotent if there exists a positive integer k such that $a^k = 0$.

Definition 2.9. A Γ-seminear-ring R is reduced if R has no non-zero nilpotent elements. We observe that as in ring theory, R has no non-zero nilpotent elements if $x\gamma x = x^2 = 0 \Rightarrow x = 0$.

We freely make use of the following notations.

(i) E = set of all idempotents of R

(ii) $C(R) = (r \in R \ / \ r\gamma x = x\gamma r$ for all $x \in R$ and $\gamma \in \Gamma)$ - centre of R.

3. $P(r,m)$ Γ-Seminear-Rings

In this section we define $P(r,m)$ Γ - seminear-rings and give certain examples of such Γ - seminear-rings.

Definition 3.1. Let r,m be two positive integers. We say that R is a $P(r,m)$ Γ - seminear-ring if $x^r\gamma R = R\gamma x^m$ for all x in R and $\gamma \in \Gamma$.

Definition 3.2. A mate function $'f'$ of R is called a P_3 mate function if for every x in R, $x\gamma f(x) = f(x)\gamma x$.

Definition 3.3. A Γ - seminear-ring R is called Boolean if $x\gamma x = x^2 = x$ for all $x \in R$, $\gamma \in \Gamma$.

Definition 3.4. An idempotent $e \in E$ is said to be a central idempotent if $e \in C(R)$.

Definition 3.5. Let I be an index set and let $(R_i)_{i\in I}$ be a family of seminear-rings. $X_{i\in I}R_i$ with the component-wise defined operations "+" and "." is called the direct product $\prod_{i\in I} R_i$ of the seminear-rings $R_i(i \in I)$.

Definition 3.6. Let R be a Γ - seminear-ring. A non-empty subset A of R is called a Sub Γ - seminear-ring if A itself is a Γ - seminear-ring with respect to the same operations as in R.

Example 3.7. (a) Let $R = \{0, a, b, c, d\}$. We define the semigroup operations $''+''$ and $''\gamma''$ in R as follows.

+	0	a	b	c	d
0	0	a	b	c	d
a	a	a	a	a	a
b	b	a	b	b	b
c	c	a	b	c	c
d	d	a	b	c	d

γ	0	a	b	c	d
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	b	c	c
d	0	a	b	c	d

Then $(R, +, \Gamma)$ is a $P(r, m)$ Γ - seminear-ring for all positive integers r and m, $\gamma \in \Gamma$.

(b) The direct product of any two seminear fields is a $P(r, m)$ Γ- seminear-ring for all positive integers r and m.

(c) The Boolean $P(1, 1)$ Γ - seminear-ring is a $P(r, m)$ Γ - seminear-ring for all positive integers r and m.

Proposition 3.8. *Any homomorphic image of a $P(r, m)$ Γ - seminear-ring is a $P(r, m)$ Γ - seminear-ring. The proof is straight forward.*

4. Properties of $P(1, 1)$, $P(1, 2)$, and $P(2, 1)$ Γ-Seminear-Rings

Proposition 4.1. *Let R be a $P(1, 2)$ Γ - seminear-ring. If R is a either right normal or left normal Γ - seminear-ring then R has no nonzero nilpotent elements.*

Proof. Since R is $P(1,2)$, $x\gamma R = R\gamma x^2$ for all $x \in R$, $\gamma \in \Gamma$. Since R is right normal, $x \in x\gamma R$. Then for every $x \in R$ we get $x = r\gamma x^2$ for some r in R, $\gamma \in \Gamma$. Thus $x^2 = 0 \Rightarrow x = r\gamma 0 = 0$. Hence R has no non-zero nilpotent elements. Proof is similar when R is left normal Γ - seminear-ring.

Definition 4.2. The Γ - seminear-ring R has strong IFP if and only if for all ideals I of R, $a\gamma b \in I \Rightarrow a\gamma x\gamma b \in I$ for $a, b \in R$ and for all $x \in R$, $\gamma \in \Gamma$.

Proposition 4.3. *If R is a $P(1,2)$ or a $P(2,1)$ Γ - seminear-ring then R has strong IFP.*

Proof. Let $a\gamma b \in I$ where I is any ideal of R and let $x \in R$, $\gamma \in \Gamma$.

Case (i) Let R be a $P(1,2)$ Γ - seminear-ring. Since I is an ideal of R, (i.e) $R\Gamma I \subseteq I$. Now $a\gamma x \in a\gamma R = R\gamma a^2 \Rightarrow a\gamma x = y\gamma a^2$ for some $y \in R \Rightarrow a\gamma x\gamma b = (a\gamma x)\gamma b = (y\gamma a^2)\gamma b = (y\gamma a)(a\gamma b) \in R\Gamma I \subseteq I \Rightarrow a\gamma x\gamma b \in I$.

Case (ii) Let R be a $P(2,1)$ Γ - seminear-ring. Since I is an ideal of R, (i.e) $I\Gamma R \subseteq I$. Now $x\gamma b\gamma R\gamma b = b^2\gamma R \Rightarrow x\gamma b = b^2\gamma y'$ for some $y' \in R \Rightarrow a\gamma x\gamma b = a\gamma(x\gamma b) = a\gamma(b^2\gamma y') = (a\gamma b)(b\gamma y') \in I\Gamma R\Gamma I \Rightarrow a\gamma x\gamma b \in I$. Hence R has strong IFP.

Proposition 4.4. *In a $P(1,2)$ Γ - seminear-ring, $E \subseteq C(R)$*

Proof. Since $0 \in E$, it is non-empty. Let $e \in E$, As R is $P(1,2)$, $e\gamma R = R\gamma e^2 \Rightarrow e\gamma R = R\gamma e \Rightarrow e\gamma R\gamma e = e\gamma(R\gamma e) = e\gamma(e\gamma R) = e^2\gamma R = e\gamma R$. Hence $e\gamma R = e\gamma R\gamma e = R\gamma e$. For $x \in R$, $\gamma \in \Gamma$ there exist $u, v \in R$ such that $x\gamma e = e\gamma u\gamma e$ and $e\gamma x = e\gamma v\gamma e$. These imply $e\gamma x\gamma e = e\gamma(x\gamma e) = e\gamma(e\gamma u\gamma e) = e\gamma u\gamma e = x\gamma e$ and $e\gamma x\gamma e = (e\gamma x)\gamma e = (e\gamma v\gamma e)\gamma e = e\gamma x$. Thus $e\gamma x = e\gamma x\gamma e = x\gamma e$ for all $x \in R, \gamma \in \Gamma$. Therefore $E \subseteq C(R)$.

Remark 4.5. The results from 4.3 and 4.4 hold good for a $P(2,1)$ Γ - seminear-ring also.

Lemma 4.6. *If R has a mate function f then R is an left (right) normal Γ - seminear-ring.*

Proof. Since R has a mate function f for all $x \in R$, $\gamma \in \Gamma$, $x = x\gamma f(x)\gamma x \in R\gamma x(x\gamma R)$. Obviously then R is a left (right) normal Γ - seminear-ring.

5. Equivalent Conditions for $P(r, m)$ Γ-Seminear-Rings

Theorem 5.1. *Let R be a Γ - seminear-ring with a mate function f. Then the following statements are equivalent.*

(i) R *is* $P(1, 2)$

(ii) $E \in C(R)$

(iii) R *is* $P(2, 1)$.

Proof. $(ii) \Rightarrow (i)$: For $a \in R$, $a\gamma x \in a\gamma R$ for all $x \in R$, $\gamma \in \Gamma$, and since $E \in C(R)$,

$$\begin{aligned} a\gamma x &= a\gamma f(a)\gamma a\gamma x = a\gamma(f(a)\gamma a\gamma x) = a\gamma x\gamma f(a)\gamma a \\ &= a\gamma x\gamma f(a)\gamma a\gamma(f(a)\gamma a) = a\gamma x\gamma f(a)\gamma(f(a)\gamma a)\gamma a \end{aligned}$$

(since $f(a)\gamma a \in E$). Therefore

$$a\gamma x = aa\gamma x\gamma f(a)^2\gamma a^2 \in R\gamma a^2 \Rightarrow a\gamma R \subseteq R\gamma a^2. \qquad (A)$$

Also

$$\begin{aligned} x\gamma a^2 \in R\gamma a^2 &\Rightarrow x\gamma a^2 = x\gamma a\gamma a = (x\gamma a)(a\gamma f(a)\gamma a) \\ &= (x\gamma a\gamma a\gamma f(a))\gamma a = a\gamma f(a)\gamma x\gamma a^2 \in a\gamma R \Rightarrow R\gamma a^2 \subseteq a\gamma R. \quad \text{(B)} \end{aligned}$$

From (A) and (B) we get $a\gamma R = R\gamma a^2$ for all a in R, $\gamma \in \Gamma$ and (i) follows.

Proof of $(i) \Rightarrow (ii)$ and that of $(iii) \Rightarrow (ii)$ are taken care of the Proposition 4.4.

$(ii) \Rightarrow (iii)$ For $a \in R$, $x\gamma a \in R\gamma a$ for all $x \in R$, $\gamma \in \Gamma$ and since $E \subseteq C(R)$,

$$\begin{aligned} x\gamma a &= x\gamma a\gamma f(a)\gamma a = (x\gamma a\gamma(f(a))\gamma a = a\gamma f(a)\gamma x\gamma a \\ &= a\gamma f(a)\gamma a\gamma f(a)\gamma x\gamma a = a\gamma a\gamma f(a)\gamma f(a)\gamma x\gamma a \end{aligned}$$

(since$f(a)\gamma a \subseteq E$)

$$x\gamma a = a^2\gamma f(a)^2\gamma x\gamma a \in a^2\gamma R \Rightarrow R\gamma a \subseteq a^2\gamma R. \qquad (C)$$

Also

$$a^2\gamma x \in a^2\gamma R \Rightarrow a^2\gamma R = a\gamma a\gamma x = a\gamma f(a)\gamma a\gamma a\gamma x = a\gamma((f(a)\gamma a)\gamma a\gamma x)$$
$$= a\gamma(a\gamma x\gamma f(a)\gamma a) = a^2\gamma x\gamma f(a)\gamma a = R\gamma a \Rightarrow a^2\gamma R \subseteq R\gamma a. \quad \text{(D)}$$

From (C) and (D) we get $R\gamma a = a^2\gamma R$ for all a in R and (iii) follows.

Remark 5.2. Let R admit a mate function f and let $E \subseteq C(R)$, we observe that for every $x \in R$, $x = x\gamma f(x)\gamma x = f(x)\gamma x^2$. Incidentally we have $x^2 = 0 \Rightarrow x = 0$. Hence R has no non-zero nilpotent elements.

Theorem 5.3. *Let R admit a mate function f. Then R is a $P(r,m)$ Γ - seminear-ring for all positive integers r and m if and only if R is a $P(1,2)$ Γ - seminear-ring.*

Proof. If part: Since R is a $P(1,2)$ Γ - seminear-ring $\Rightarrow$ $E \subseteq C(R)$ (by Proposition 4.4) Let r, m be any two positive integers. Let $a \in x^r\gamma R$. Therefore $a = x^r\gamma y$ for some y in R. Now $a = (x\gamma f(x)\gamma x)^r\gamma y = x^r\gamma(f(x)\gamma x)^r\gamma y$ (since $f(x)\gamma x \in E \subseteq C(R)$)$= x^r\gamma(f(x)\gamma x)\gamma y = x^r\gamma y\gamma f(x)\gamma x$ (since $E \subseteq C(R)$) $=$ $x^r\gamma y\gamma(f(x))^m x^m$ (since $f(x)\gamma x \in E$)$= x^r\gamma y\gamma(f(x))^m x^m$ (since $E \subseteq C(R)$) $=$ $(x^r\gamma y\gamma(f(x))^m)\gamma x^m \in R\gamma x^m$. Therefore $x^r\gamma R \subseteq R\gamma x^m$. In a similar fashion we get $R\gamma x^m \subseteq x^r\gamma R$. Hence $x^r\gamma R = R\gamma x^m$ and R is a $P(r,m)$ Γ - seminear-ring. The converse is obvious - it follows by taking $r = 1$ and $m = 2$. We furnish below a characterization of $P(r,m)$ Γ - seminear-rings.

Theorem 5.4. *Let R be a Γ - seminearring with a mate function f. Then R is $P(r,m)$ if and only if for every $x \in R$, $\gamma \in \Gamma$, there exists a central idempotent e such that $R\gamma x = R\gamma e$.*

Proof. For the only if part, let $x \in R$,$\gamma \in \Gamma$. Then $R\gamma x = R\gamma f(x)\gamma x = R\gamma e$ where $e = f(x)\gamma x \in E$. But in a $P(r,m)$ Γ- seminear-ring $E \subseteq C(R)$ (by Proposition 4.4). Therefore $R\gamma x = R\gamma e$ where e is a central idempotent. For the if part, we need only to show that $E \subseteq C(R)$ (in view of Theorems 5.1 and 5.3). Let $e_1 \in E$. Now $R\gamma e_1 = R\gamma e$ for some central idempotent e. Now $e_1 = e_1^2 \in R\gamma e_1(= R\gamma e) \Rightarrow e_1 = y\gamma e$ for some $y \in R$, $\gamma \in \Gamma$. Therefore

$$e_1 = (y\gamma e)\gamma e = e_1\gamma e. \quad (1)$$

Also $e = e^2 \in R\gamma e(= R\gamma e_1) \Rightarrow e = u\gamma e_1$ for some $u \in R$, $\gamma \in \Gamma$. Therefore

$$e = u\gamma e_1 = (u\gamma e_1)\gamma e_1 = e\gamma e_1. \quad (2)$$

Since $'e'$ is a central idempotent

$$e\gamma e_1 = e_1\gamma e. \tag{3}$$

From (1), (2) and (3) we get $e_1 = e_1\gamma e = e\gamma e_1 = e$. Therefore $e_1(= e)$ is a central idempotent. Thus $E \subseteq C(R)$. Therefore R is a $P(r, m)$ Γ - seminear-ring.

References

[1] Balakrishnan R. and Perumal R., Left Duo Seminear-rings, *Scientia Magna*, **8(3)**, 115 -120, (2012).

[2] Booth G. L. and Groenewald N. J., On strongly prime near-rings, *Indian Journal of Mathematics*, **40(2)**, 113 - 121, (1998).

[3] Jat J.L. and Choudary S.C., On left bipotent near-rings, *Proc. Edin. Math. Soc.*, **22**, 99 - 107, (1979).

[4] Javed AHSAN, Seminear-rings Characterized by their s-ideals. I *Proc. Japan Acad*, **71(A)**, 101 - 103, (1995).

[5] Javed AHSAN, Seminear-rings Characterized by their s-ideals. II *Proc. Japan Acad*, **71(A)**, 111 - 113, (1995).

[6] Park. Y. S and Kim. W. J, On Structures of Left Bipotent Near-rings, *Kyungbook Math. J,* **20(2)**, 177 - 181, (1980).

[7] Pellegrini Manara S., On regular medial near-rings,*Boll. Unione Mat. Ital., VI Ser., D, Algebra Geom*, **6**(1985), 131 - 136.

[8] Pellegrini Manara S., On medial near-rings, Near-Rings and Near fields, Amsterdam, North Holland, 199 - 210, (1987).

[9] Perumal R., Balakrishnan R. and Uma S., Some Special Seminear-ring Structures, *Ultra Scientist of Physical Sciences.*, **23(2)**, 427 - 436, (2011).

[10] Perumal R., Balakrishnan R. and Uma S., Some Special Seminear-ring Structures II, *Ultra Scientist of Physical Sciences.*, **24(1)**, 91 - 98, (2012).

[11] Perumal R. and Balakrishnan R., Left Bipotent Seminear-rings, *International Journal of Algebra.*, **6(26)**, 1289 -1295, (2012).

[12] Pilz Günter, Near-Rings, North Holland (1983), Amsterdam.

[13] Shabir. M and Ahamed. I, Weakly Regular Seminearrings *International Electronic Journal of Algebra*,**2**(2007), 114 - 126.

[14] Weinert H.J., Seminear-rings, Seminearfield and their Semigroup Theoretical Background, *Semigroup Forum*, **24**, 231 - 254, (1982).

[15] Willy G. van Hoorn and van Rootselaar R., Fundamental notions in the theory of Seminearrings, *Compositio Math*, **18**, 65 - 78, (1967).

[16] Weinert H.J., Related Representation theorems for Rings, Semi-rings, Near-rings and Seminear-rings by Partial Transformations and Partial Endomorphisms, *Proceedings of the Edinburgh Mathematical Society*, **20**, 307 - 315, (1976-77).